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THE ANALYTICAL GEOMETRY OF THE CONIC SECTIONS

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PREFACE TO THIRD EDITION

THIS Edition is substantially the same as the last with a few minor corrections. The references to my *Course of Pure Geometry* (Cambridge University Press) are to the Enlarged Edition of 1917, which has been several times reprinted, but with only slight changes which leave the pages and paragraphs as they were.

E. H. A.

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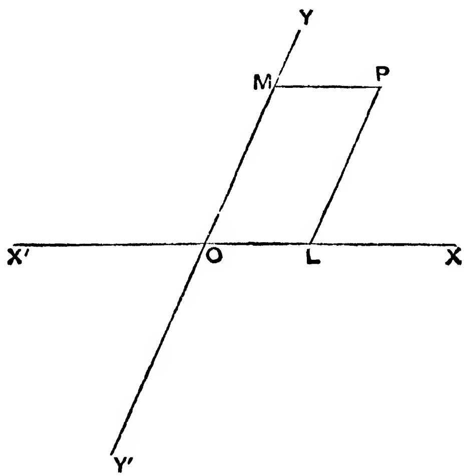
CHAPTER I.

CARTESIAN AND POLAR COORDINATES.

1. Cartesian Coordinates. Let XOX', YOY' be two given intersecting lines, P any point in their plane. Let the parallelogram $PLOM$ be completed having its adjacent sides OL, OM along the given lines XOX', YOY' respectively. The position in the plane of the point P relatively to the given lines is known when the magnitudes and directions of OL and OM are known.



It is necessary in order that the point P may be definitely determined that the directions as well as the magnitudes of OL and OM should be known. For if only the magnitudes were given we should not know whether OL was to be

measured in the direction OX or in the direction OX', nor again whether OM was to be measured in the direction OY or in the direction OY'.

Accordingly we make use of that convention of signs with which the reader is already acquainted in Trigonometry, and consider OL to be a positive magnitude if it is in the direction OX, and negative if it is in the opposite direction OX'. OM too is to be considered positive if in the direction OY, and negative in the direction OY'.

OL and OM, regard being had to their sign as well as to their magnitude, are called *the Cartesian coordinates of the point P with respect to the axes OX, OY.* They are so called after the celebrated French mathematician, des Cartes, who first introduced this method of determining the position of a point in a plane.

By calling the axes of coordinates OX, OY we imply that OX, OY are the directions in which OL, OM, as explained above, are to be accounted positive.

We distinguish the two axes of coordinates by calling OX the x-axis, and OY the y-axis. O is called the *origin.*

OL may be called the x-coordinate of P, OM the y-coordinate. OL is frequently denoted by x, and OM by y. The point P is then briefly represented as (x, y).

When we wish to represent several points we can do so by means of suffixes. Thus (x_P, y_P) can represent P, and (x_Q, y_Q) can represent a different point Q. Different points are sometimes represented by means of numerical suffixes as (x_1, y_1), (x_2, y_2), (x_3, y_3), &c., or again by dashes, as (x', y'), (x'', y''), (x''', y'''), &c.

2. It will be convenient to speak of that part of the plane which falls within the angle XOY (OX, OY being unlimited), as the *first quadrant,* the part within the angle YOX' as the *second quadrant,* the part within $X'OY'$ as the *third quadrant,* and the part within $Y'OX$ as the *fourth quadrant.*

If then a point lie in the first quadrant both its x and y coordinates are positive; if in the second quadrant its x-coordinate is negative and its y-coordinate positive; if in the third quadrant both coordinates are negative; if in the fourth quadrant the x-coordinate is positive and the y-coordinate negative.

To represent the point $(3, -4)$ in the figure, OL would have to be measured along OX of 3 units of length, and OM along OY' of 4 units of length; the parallelogram $OLPM$ would then have to be completed. The vertex P is the point $(3, -4)$.

The student may as an exercise mark on properly ruled graph paper the following points, $(2, 1)$, $(8, -3)$, $(-4, 5)$, $(5, -1)$. These points, if correctly marked, will be found to lie in one straight line.

3. The relativity of coordinates. The student cannot too early familiarise himself with the fact that the coordinates of a point are purely *relative.* That is, they depend for their sign and magnitude on the axes of coordinates. For different axes the same point will have different coordinates.

The following is of great importance:

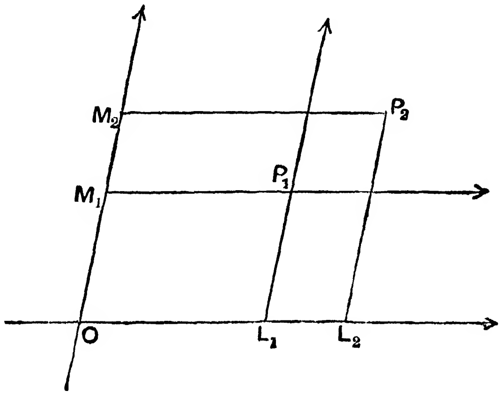
If (x_1, y_1) (x_2, y_2) be the coordinates of two points P_1 and P_2 relative to axes OX and OY, then the coordinates of P_2 *relative to axes through* P_1 *parallel to* OX, OY will be $(x_2 - x_1, y_2 - y_1)$. For if we complete the parallelograms $OL_1P_1M_1$, $OL_2P_2M_2$, we see that the coordinates of P_2 relative to the new axes through P_1 are equal to L_1L_2 and M_1M_2.

But
$$L_1L_2 = OL_2 - OL_1 = x_2 - x_1,$$
$$M_1M_2 = OM_2 - OM_1 = y_2 - y_1.$$

This is true in whatever quadrant the points P_1 and P_2 may happen to fall, whether the same or different ones.

The student may mark the points P_1 and P_2 whose coordinates are $(-3, 5)$, $(1, -7)$ and satisfy himself that the coordinates of P_2 relative to axes through P_1, parallel to the

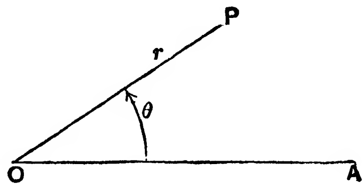
original axes, are $4\ \{=1-(-3)\}$, and $-12\ \{=-7-5\}$, and that the coordinates of P_1 relative to axes through P_2 parallel to the original axes are $(-4, 12)$.



4. Rectangular and oblique axes. When the axes of coordinates are at right angles they are said to be *rectangular*, otherwise they are called *oblique*. As the student will see, rectangular axes are used more commonly than are oblique. There are cases, however, where oblique axes are useful. The student will find it necessary as he goes along to discriminate formulae which are applicable only when the axes are rectangular and formulae which hold for oblique axes as well.

5. Polar coordinates. In this system of coordinates the position of a point is determined by its distance from a fixed point O, usually called the *pole* (though it might equally well be called the origin), and the angle which the line joining the pole to the point makes with a fixed line through the pole, called the *initial line*. Thus if OA be the initial line, the polar coordinates of a point P are OP which is known as the *radius vector*, and the angle AOP which is called the *vectorial angle*. The vectorial angle is measured from the initial line as in Trigonometry; it is usually considered positive if measured round from OA in the opposite direction to that of the rotation

of the hands of a watch, and negative in the other direction. But it may on occasion be more convenient to take the rotation positive in the same direction as that of the hands of a watch. To mark a point whose polar coordinates (r, θ) are given, we first measure the vectorial angle θ and then cut off the radius vector $(=r)$. The extremity P of this is the point (r, θ).



6. On the sign of the radius vector. In polar coordinates we admit negative as well as positive radii vectores, a negative radius vector being measured in the opposite direction to that of the boundary line of the vectorial angle.

It will be seen that the point whose polar coordinates are $(-c, \alpha)$ is the same as the points $(c, \pi+\alpha)$. It may seem then to be unnecessary to admit negative radii vectores at all, since every point with a negative radius vector could be equally well represented by means of a positive one by a change in the vectorial angle.

We cannot, however, afford to exclude the negative radius vector; for while points in isolation can be as well expressed by means of a positive radius vector as by a negative one, this is not the case when we have to do with an assemblage of points forming a curve or locus, as it is called.

We can illustrate this point by a simple example. There is a certain assemblage of points forming a curve whose polar coordinates satisfy the equation

$$\frac{1}{r}=1+3\cos\theta.$$

Now it can be seen that for $r=-2$, $\theta=\frac{2\pi}{3}$ this equation is satisfied, and so we say that the curve represented by this

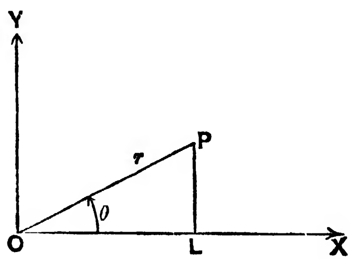
equation passes through the point $\left(-2, \frac{2\pi}{3}\right)$. But the coordinates of this point when it is expressed by means of a positive radius vector as $\left(2, -\frac{\pi}{3}\right)$ do not satisfy the equation. We should then have to exclude this point in the plane as not part of the locus if we admitted only positive radii vectores, whereas by admitting the negative radius vector the point belongs to the locus, as it will be required to do. Why it will be so required it is premature to attempt to explain to the reader. He will be in a position to understand this for himself when he has mastered the polar equation of conics.

7. Formulae connecting the polar and Cartesian coordinates of a point.

It is to be understood in what follows that the pole and the initial line in the polar system are respectively the origin and the axis of x in the Cartesian system, and the positive direction of measurement of the vectorial angle is towards the axis of y.

Let (x, y) be the Cartesian coordinates of a point P, (r, θ) its polar coordinates.

First, let the Cartesian axes be rectangular.



We have at once from the figure

$$x = r\cos\theta, \quad y = r\sin\theta,$$

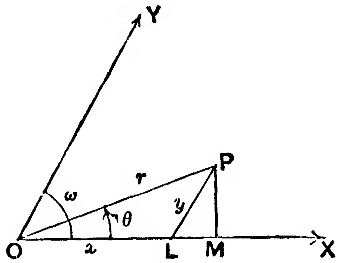
and these formulae hold in whichever quadrant P may be.

From the above we have

$$r^2 = x^2 + y^2,$$

$$\tan\theta = \frac{y}{x}.$$

Secondly, let the Cartesian axes be oblique.



Drawing PL parallel to the axis of y to meet the x-axis in L, and PM perpendicular to the x-axis, we have, if ω be the angle between the axes,

$$r\cos\theta = OM = OL + LM = x + y\cos\omega,$$
$$r\sin\theta = MP = y\sin\omega.$$

Adding squares we have

$$r^2 = (x + y\cos\omega)^2 + y^2\sin^2\omega$$
$$= x^2 + 2xy\cos\omega + y^2.$$

This is an important formula giving as it does the distance of the point (x, y) from the origin.

8. Distance between two points whose Cartesian coordinates are given.

Let (x_1, y_1), (x_2, y_2) be the coordinates of the points P_1 and P_2 respectively. The coordinates of P_2 relatively to axes through P_1 parallel to the original axes are $x_2 - x_1$, $y_2 - y_1$.

Therefore by the last article

$$P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega,$$

or, what is the same thing,

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega.$$

In the case where the axes are rectangular

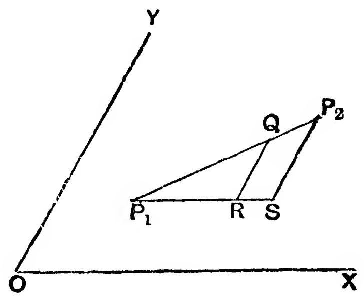
$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

9. Coordinates of the point dividing in a given ratio the line joining two given points.

Let (x_1, y_1) (x_2, y_2) be the coordinates of the given points P_1 and P_2.

Let Q be a point in P_1P_2 such that $P_1Q : QP_2 = k : l$

Let (x, y) be the coordinates of Q.



Draw QR and P_2S parallel to the y-axis to meet the line through P_1 parallel to the x-axis in R and S.

Then $$k : l = P_1Q : QP_2 = P_1R : RS$$
$$= x - x_1 : x_2 - x$$
$$\therefore\ k(x_2 - x) = l(x - x_1),$$
$$\therefore\ x = \frac{kx_2 + lx_1}{k + l}.$$

Similarly $$y = \frac{ky_2 + ly_1}{k + l}.$$

Cor. If Q be the middle point of P_1P_2 its coordinates are $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$.

It should be noted that if Q does not lie between P_1 and P_2 then the ratio $k : l$ is a negative quantity, that is to say one of the two, k or l, is negative. It does not make any difference to which of the two the negative sign be given.

Area of a triangle.

10. *To find the area of the triangle formed by joining the origin to two points P_1, P_2 whose coordinates are (x_1, y_1), (x_2, y_2).*

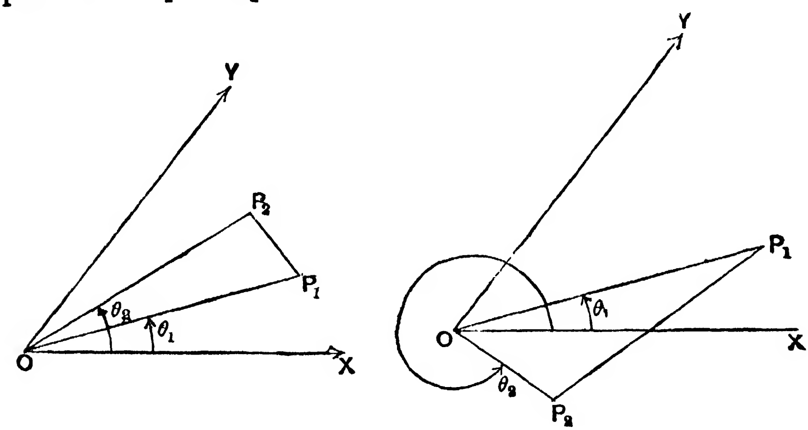
Let (r_1, θ_1), (r_2, θ_2) be the polar coordinates of P_1 and P_2. Let $\theta_2 > \theta_1$.

$$\text{The area of the triangle } OP_1P_2 = \tfrac{1}{2}\, OP_1 . OP_2 \sin P_1OP_2$$

$$= \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1),$$

or $$\tfrac{1}{2} r_1 r_2 \sin\{2\pi - (\theta_2 - \theta_1)\},$$

according as the origin is to the left or right hand as we pass from P_1 to P_2.



Thus the area of the triangle OP_1P_2

$$= \pm \tfrac{1}{2} r_1 r_2 (\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1)$$

$$= \pm \tfrac{1}{2} \{(x_1 + y_1 \cos\omega)\, y_2 \sin\omega - (x_2 + y_2 \cos\omega)\, y_1 \sin\omega\} \text{ by § 7}$$

$$= \pm \tfrac{1}{2} (x_1 y_2 - x_2 y_1) \sin\omega.$$

If we consider the triangle OP_1P_2 to have a positive area when, as we proceed round the triangle from O to P_1 and then to P_2, the triangle is on our left hand, and to have a negative area in the opposite case, then

$$\triangle\, OP_1P_2 = -\triangle\, OP_2P_1,$$

and we have $$\triangle\, OP_1P_2 = \tfrac{1}{2}(x_1 y_2 - x_2 y_1) \sin\omega,$$

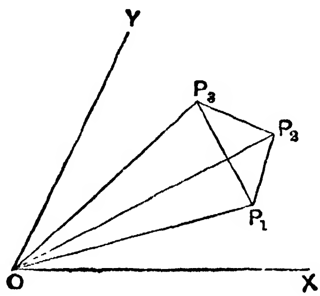
which when the axes are rectangular $= \frac{1}{2}(x_1 y_2 - x_2 y_1)$.

11. *To find the area of the triangle whose vertices are* (x_1, y_1), (x_2, y_2), (x_3, y_3).

Denoting these points by P_1, P_2, P_3 respectively, we see that the coordinates of P_2 relatively to P_1 are $(x_2 - x_1, y_2 - y_1)$ and of P_3 relatively to P_1 $(x_3 - x_1, y_3 - y_1)$.

Therefore by the last article, with the same convention as to sign,

$$\triangle P_1P_2P_3 = \tfrac{1}{2} \sin \omega \{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\}$$
$$= \tfrac{1}{2} \{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \sin \omega.$$



This is a formula easily remembered. It can be written as a determinant thus:

$$\tfrac{1}{2} \sin \omega \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} \quad \text{or} \quad \tfrac{1}{2} \sin \omega \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

12. The result of the previous article could also be obtained by joining the vertices P_1, P_2, P_3 to the origin O. Then regard being had to the signs of the areas

$$\triangle P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1$$
$$= \tfrac{1}{2} \sin \omega \{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\}.$$

And in this way we can obtain the area of a polygon $P_1P_2P_3 \ldots P_n$.

For the polygon

$$= \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_4 + \triangle OP_4P_5 + \ldots + \triangle OP_{n-1}P_n + \triangle OP_nP_1$$

$$= \tfrac{1}{2} \sin \omega \left\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + \ldots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\right\}.$$

13. Condition of collinearity of three points. The points (x_1, y_1), (x_2, y_2), (x_3, y_3) will clearly lie on one straight line if the area of the triangle formed by joining the points is zero. The condition for this is

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0,$$

or

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} = 0.$$

This condition for collinearity of three points holds whether the axes be rectangular or oblique.

EXAMPLES.

1. Prove that the distance between two points whose polar coordinates are (r_1, θ_1), (r_2, θ_2) is

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

2. Write down the coordinates of the middle points of the sides of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), and shew that the area of the triangle formed by joining these is one-quarter of that of the original triangle.

3. Find the distance between the points $(2, -3)$, $(-5, -7)$, the axes being inclined at 60°.

4. Shew that the points (a, a), $(-a, -a)$, $(-a\sqrt{3}, a\sqrt{3})$ are the vertices of an equilateral triangle, the axes being rectangular.

5. Shew that the points $(1, -\frac{3}{2})$, $(-3, -\frac{7}{2})$, $(-4, -\frac{3}{2})$ are the vertices of a right-angled triangle, the axes being rectangular.

6. Shew that the points (2, 3), (6, 9) are in a straight line with the origin.

Shew that (a, b), (ka, kb) are in a straight line with the origin.

7. Prove that the three points $(1, \frac{7}{3})$, $(2, \frac{5}{3})$, $(5, -\frac{1}{3})$ are collinear.

8. If (x, y) be any point in the straight line which passes through (2, 4) and (5, 9), prove that

$$5x - 3y + 2 = 0.$$

[Express the fact that the area of the triangle with the three points as vertices is zero.]

9. Prove that the points (x_1, y_1), (x_2, y_2), $\left(\frac{kx_2 + lx_1}{k+l}, \frac{ky_2 + ly_1}{k+l}\right)$ are collinear by § 13.

10. If the point (x, y) be equidistant from the points $(2, -3)$ and $(-5, -7)$ then $14x + 8y + 61 = 0$, and conversely, the axes of coordinates being rectangular.

11. Shew that the middle point of the line joining (5, 1) and (3, 7) is also the middle point of the line joining (20, 9) and $(-12, -1)$. What geometrical conclusion can be drawn from this fact?

12. Find the area of the quadrilateral whose vertices are $(4, 5)$, $(6, -2)$, $(3, 8)$, $(5, 1)$, the axes being inclined at 30°.

[Care must be taken that the points are taken in the proper order so as to get a closed figure.]

CHAPTER II.

LOCI AND THEIR EQUATIONS.

14. In the preceding chapter we have explained the method of representation of isolated points in a plane by means of Cartesian and Polar coordinates. We pass now to consider the representation of an assemblage of points. We cannot represent analytically an assemblage of points taken at random, but when the points are situated according to some law it may be possible to give a collective representation of them. Thus all points which lie on the same straight line can be represented by means of an equation.

For example, if (x, y) be the coordinates of *any* point on the line passing through the points (2, 4) and (3, 5) then the area of the triangle formed by these points is zero. We thus have

$$\begin{vmatrix} x, & y, & 1 \\ 2, & 4, & 1 \\ 3, & 5, & 1 \end{vmatrix} = 0,$$

from which we get $x - y + 2 = 0$.

Now this equation is satisfied by the x and y coordinates of *every* point on the line. It may therefore be said to *represent* the line, and it is called the *equation of the line.*

15. Again, we might find an equation satisfied by the coordinates of all points which are at a certain given distance from a point whose coordinates are given. Thus if (x, y) be the coordinates of any point whose distance from the point (2, 3) is 5 we have, if the axes be rectangular,

$$(x-2)^2 + (y-3)^2 = 5^2,$$

for the left-hand side of this equation is (§ 8) the square of the distance between the points (x, y) and $(2, 3)$. This equation reduces to

$$x^2 + y^2 - 4x - 6y = 12.$$

This equation then represents a circle whose centre is at $(2, 3)$ and whose radius is 5. And we speak of the equation as the *equation of this circle.*

16. A number of points obeying some law are said to form a *locus,* and if it be possible to find an equation satisfied by all points which obey that law, but satisfied by no other points, that equation is called the equation of that locus. Thus the circumference of a circle having its centre at the point (h, k) and having its radius equal to a is the locus of points whose distance from (h, k) is a. Its equation is then

$$(x - h)^2 + (y - k)^2 = a^2,$$

for this equation is plainly satisfied by all points whose distance from (h, k) is a, and by no other points.

17. Points which form a locus for which an equation exists lie along a line straight or curved. But a locus is not necessarily a straight or curved line. For example, all points lying within the circumference of some particular circle form a locus, and as we shall see in a later chapter, they can be represented collectively, though not by an equation. Every locus, however, which can be represented by an equation may be called a *curve,* by which is meant the line on which all the points which obey the law of the locus lie, and on which lie no points which do not belong to the locus.

18. The relativity of the equations of curves or loci. It is clear that the equation of a curve or locus is relative to the axes of coordinates, and it will change when the axes of coordinates are changed. Thus the equation of a circle whose centre is at (h, k) and whose radius is a is, as we have seen,

$$(x - h)^2 + (y - k)^2 = a^2 \quad \text{..................(1).}$$

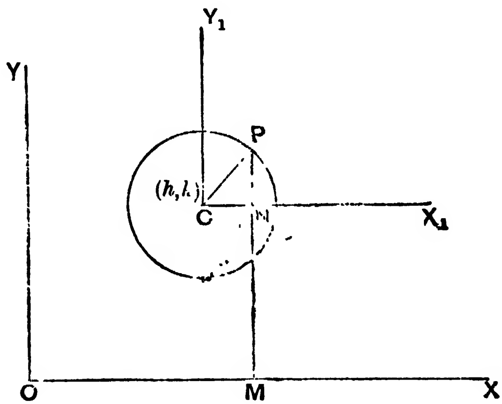
But if the axes of coordinates pass through the centre of the circle the equation of this same circle becomes

$$x^2 + y^2 = a^2 \quad \text{.......................}(2).$$

The x and y of equation (2) are of course not the same as the x and y of equation (1). In (2) they are the coordinates of a point P on the circle with reference to the axes through its centre, but in (1) they are the coordinates of the same point P, but with reference to different axes.

Thus in the accompanying figure which, it is hoped, is self-explanatory, the x and y of equation (1) are OM and MP respectively; but the x and y of equation (2) are CN and NP respectively, Both the equations alike represent the geometrical fact that

$$CN^2 + NP^2 = CP^2 = a^2.$$



19. If we wish to discover by analytical methods the geometrical properties of a curve or locus whose equation can be found, we choose our axes of coordinates so as to make the equation of the curve as simple as possible. Not that the same properties could not be proved otherwise, but the working out of the proof would be more complicated. The student will learn by experience which axes it will be best to take in particular cases.

EXAMPLES.

[*The axes of coordinates are to be taken rectangular.*]

1. Find the equation of the locus of points which are equidistant from the points (2, 3) and (5, 7).

2. Find the equation of the locus of points whose distance from the point (2, −3) is double their distance from (1, 2).

[Here

$$\sqrt{(x-2)^2+(y+3)^2} = 2\sqrt{(x-1)^2+(y-2)^2}.$$

Square and simplify.]

3. Find the equation of the locus of a point whose distance from the origin is twice its distance from the axis of x.

4. Find the equation of the locus of a point whose distance from $(a, 0)$ is equal to its distance from the axis of y.

5. Find the equation of the locus of a point whose distance from $(a, 0)$ is m times its distance from $(0, a)$.

6. Find the equation of the locus of points the sum of the squares of whose distances from the points (2, 5), (3, −1) is equal to 40.

7. A point moves in its plane so that the sum of its distances from the points $(c, 0)$, $(-c, 0)$ is $2a$. Show that the equation of its locus is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b^2=a^2-c^2$.

8. Express analytically the locus of points whose distances from two given points are in a given ratio k.

[Here the axes of coordinates are at our choice. Let A and B be the fixed points. Let $AB=2a$. Take O the middle point of AB for origin, OA and a line perpendicular to it for axes of coordinates. The equation of the locus will be found to be

$$(1-k^2)(x^2+y^2)-2ax(1+k^2)+a^2(1-k^2)=0.]$$

9. Express analytically the locus of a point the sum of the squares of whose distances from two given points is constant.

10. The equation of a certain curve is

$$x^2+y^2-4x+6y=14.$$

What will this become when the origin is transferred to (2, −3) without changing the directions of the axes?

CHAPTER III.

THE STRAIGHT LINE.

20. Equation of a line through two given points.

To find the equation of the line passing through the points A and B whose coordinates are (x_1, y_1), (x_2, y_2) *respectively.*

Let (x, y) be the coordinates of any point P on the line.

The coordinates of P relative to axes through A and parallel to the given axes are $(x - x_1, y - y_1)$; and the coordinates of B relative to the same axes are $(x_2 - x_1, y_2 - y_1)$.

Thus the area of the triangle PAB is

$$\tfrac{1}{2} \sin \omega \{(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)\},$$

where ω is the angle between the axes.

But this area is zero since P, A and B are collinear.

Therefore $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad \text{......................(A).}$$

This then is the equation of the line passing through the two given points.

By changing the signs of both denominators we can write this also

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \quad \text{......................(A).}$$

And it can easily be seen that the equation is equivalent to

$$\frac{x - x_2}{x_1 - x_2} = \frac{y - y_2}{y_1 - y_2} \quad \text{......................(A),}$$

as of course the line through (x_1, y_1) and (x_2, y_2) is also the line through (x_2, y_2) and (x_1, y_1). In other words the equation must be unaltered when we interchange x_1 and x_2, y_1 and y_2, simultaneously.

21. Line through the origin. In the special case where the point (x_2, y_2) is the origin we have $x_2 = 0$, $y_2 = 0$ and thus the equation of the line through the origin and (x_1, y_1) is

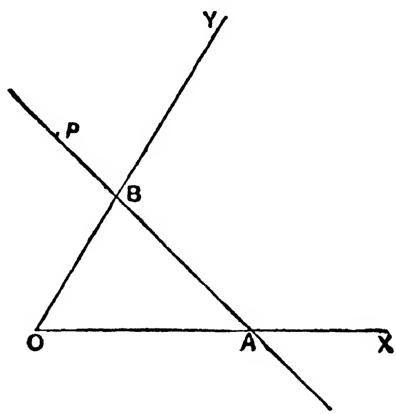
$$\frac{x}{x_1} = \frac{y}{y_1},$$

which may be written $$\frac{y}{x} = \frac{y_1}{x_1}.$$

In other words the ratio of the x and y coordinates of any point on the line is constant, as is obvious also from geometrical considerations.

22. Equation of line whose intercepts on the axes are given.

Let ABP be the line cutting the axes in A and B. Let $OA = a$, $OB = b$ where a and b may be one or other or both negative.



Let (x, y) be the coordinates of any point on the line. Then the area of the triangle whose vertices are (x, y), $(a, 0)$, $(0, b)$ is zero.

Therefore $x(0-b)+a(b-y)+0(y-0)=0$;

$$\therefore\ bx+ay=ab;$$

$$\therefore\ \frac{x}{a}+\frac{y}{b}=1 \quad \text{......................(B).}$$

This is an important form and one easily remembered.

The equation (B) could of course be derived from (A) by writing $x_1=a,\ y_1=0;\ x_2=0,\ y_2=b$;

$$\therefore\ \frac{x-a}{a-0}=\frac{y-0}{0-b};$$

$$\therefore\ \frac{x}{a}+\frac{y}{b}=1.$$

It should be noticed that both equations (A) and (B) are applicable whether the axes be rectangular or oblique.

23. General linear equation.

To prove that the equation $Ax+By+C=0$ where A, B, C are constants represents a straight line.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be three points on the locus represented by the given equation, thus

$$Ax_1+By_1+C=0,$$
$$Ax_2+By_2+C=0,$$
$$Ax_3+By_3+C=0.$$

Multiply the first equation by (y_2-y_3), the second by (y_3-y_1), and the third by (y_1-y_2), then by addition

$$A\,[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]=0$$

for the terms in B and C vanish identically.

Therefore if A be not zero

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0,$$

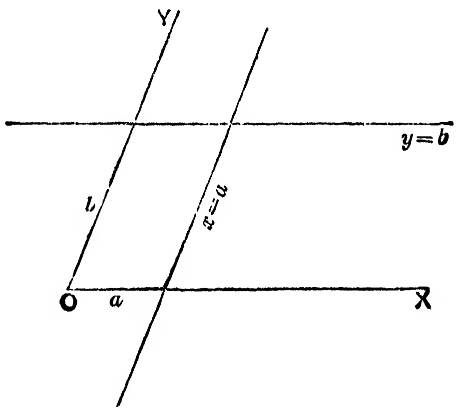
that is, the area of the triangle formed by joining the three points is zero; therefore the three points lie on one straight line. This being true of *any* three points on the locus, the locus itself is a straight line.

If A be zero, the equation reduces to $By + C = 0$, that is

$$y = -\frac{C}{B}.$$

Thus the y coordinate of every point on the locus is the same. The locus is then a straight line parallel to the axis of x.

24. Lines parallel to the axes. It is easily seen from what has been said at the conclusion of the last article, that the equation of a line parallel to the x-axis is of the form $y = b$, and of a line parallel to the y-axis $x = a$.



These are special cases of the form (B) when one or other of the intercepts on the axes becomes infinite.

Examples. 1. Use (A) to obtain the equation of the line through the points (6, 7), (3, 1) in the form $2x - y = 5$.

2. Find the equation of the line through (2, 4), $(-3, 6)$.

3. Write down the equation of the line through (1, 3), $(-2, -6)$ and shew that it passes through the origin.

4. Shew that the equation of the line through $\left(-\frac{c}{m}, 0\right)$, $(0, c)$ is $y = mx + c$.

5. Shew that the equation of the line through $(p \sec a, 0)$, $(0, p \operatorname{cosec} a)$ is $x \cos a + y \sin a = p$.

6. Find the intercepts on the axes made by the line $2x+5y=7$.

[Write the equation $\dfrac{x}{\frac{7}{2}}+\dfrac{y}{\frac{7}{5}}=1$.

Compare with (B) and the intercepts are seen to be $\frac{7}{2}$, $\frac{7}{5}$.

Or in the equation $2x+5y=7$, put $y=0$,

$$\therefore\ 2x=7,\quad \therefore\ x=\tfrac{7}{2},$$

this then is the intercept on the x-axis.

Again, put $x=0$ and we get $y=\frac{7}{5}$ which is the intercept on the y-axis.]

7. Find the intercepts on the axes made by the line $5x-4y=20$ and so draw the line.

Forms for the equation of a straight line when the axes are rectangular.

25. Thus far we have made no restriction as to the axes of coordinates. The formulae (A) and (B) are good for oblique axes as for rectangular axes. So also it is true that

$$Ax+By+C=0$$

represents a straight line when the axes are oblique as well as when they are rectangular. But we are now going to obtain *special forms of the equation of a straight line when the axes are rectangular.* These will be of constant use hereafter.

26. *To obtain the equation of a straight line in terms of the perpendicular from the origin upon it, and the angle which this perpendicular makes with the axis of x, the axes being supposed rectangular.*

Let OL the perpendicular from the origin be p and the angle XOL be α.

Then if OA, OB be the intercepts on the axes

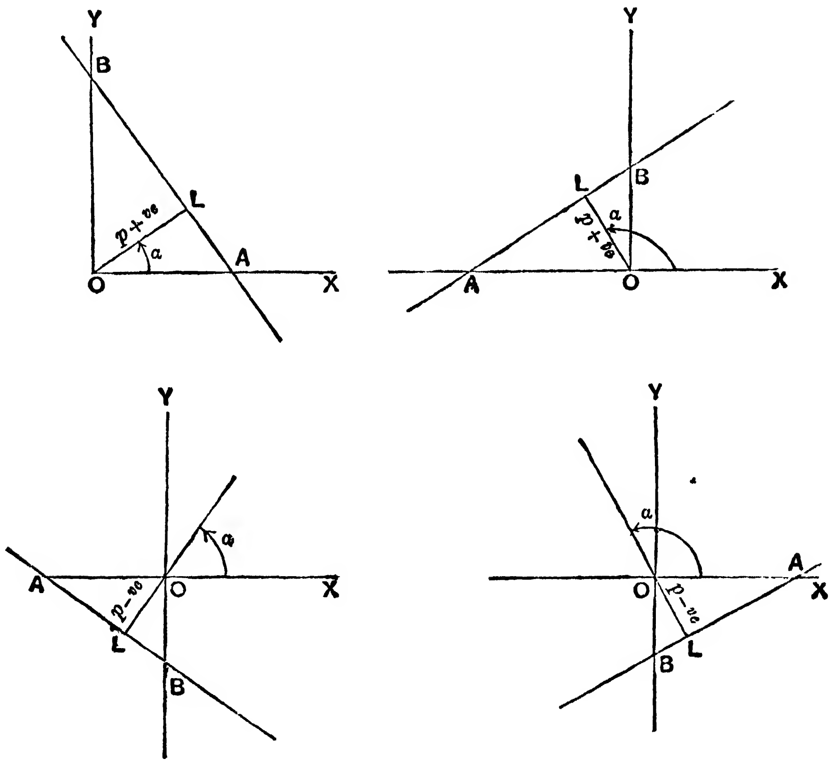
$$OA=p\sec\alpha,$$

$$OB=p\operatorname{cosec}\alpha.$$

Therefore the equation of the line is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1,$$

that is, $$x \cos \alpha + y \sin \alpha = p \quad \text{.................(C).}$$



This is a very important form for the equation of a line, and it holds good however the line falls, provided that α is measured only from 0 to π in the positive direction, and provided that the perpendicular from the origin be accounted positive in the first two quadrants and negative in the third and fourth.

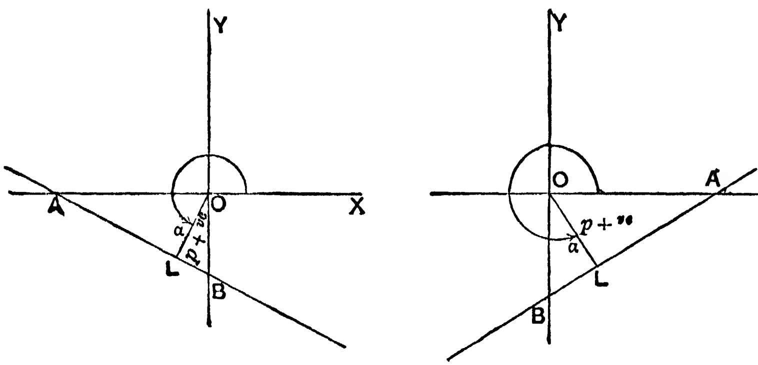
The student should carefully verify for himself that in each of the cases represented in the accompanying figures the intercepts on the axes are $p \sec \alpha$ and $p \operatorname{cosec} \alpha$ respectively.

The point is that by making our convention in regard to the measurement of α and the sign of p, we ensure that the intercepts have their right signs.

27. The student may satisfy himself that the form

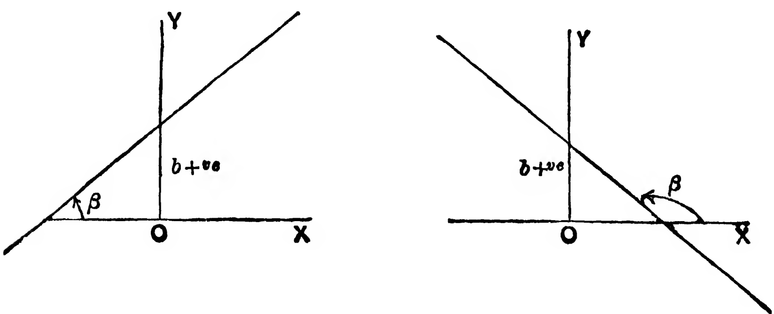
$$x \cos \alpha + y \sin \alpha = p$$

holds in all cases equally well if α be measured from 0 to 2π and p be always accounted positive.



28. *To find the equation of a straight line whose inclination β to the x-axis is known, as also its intercept b on the y-axis.*

It will be seen from the figures that in each case the intercepts on the axes are $-b \cot \beta$ and b respectively.

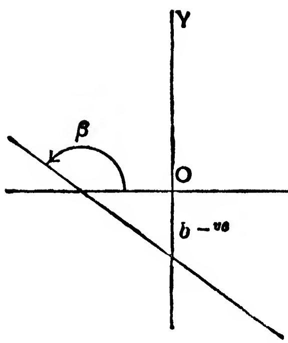


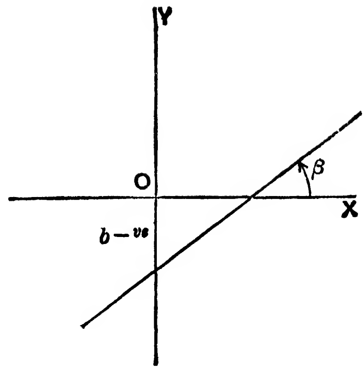
Therefore the equation of the line is

$$\frac{x}{-b\cot\beta}+\frac{y}{b}=1,$$

or

$$y=x\tan\beta+b.$$





It is usual to write m for $\tan\beta$. That is m stands for the tangent of the inclination of the line to the axis of x. We then have the standard form for the equation of a line

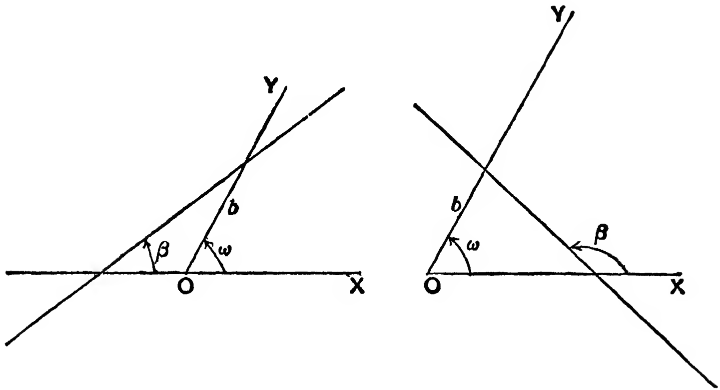
$$y=mx+b \quad \text{........................(D).}$$

It will be observed that b has the same meaning as in (B). Some writers use c for b in (D). It really does not matter what letter is used. The advantage of using the same letter in (B) and (D) is that attention is drawn to the fact that the same thing is represented each time.

29. It should be noticed at this point that the equation of a line can still be written in the form $y=mx+b$ when the axes are oblique, b being the intercept on the y-axis. *But m is not now the tangent of the angle which the line makes with the axis of x,* but if β be the angle the line makes with the axis of x, and ω the angle between the axes, then

$$m=\frac{\sin\beta}{\sin(\omega-\beta)}.$$

This follows at once from the fact that the intercept on the x-axis is $b\dfrac{\sin(\beta-\omega)}{\sin\beta}$ or, what is the same, $-b\dfrac{\sin(\omega-\beta)}{\sin\beta}$.



30. On the constants in the equation of a line. It will be observed that when the equation of a line is written in the form $y = mx + b$, of the two constants m and b which occur, m depends on the direction of the line, and b on its position.

The two lines then whose equations are

$$y = mx + b,$$
$$y = mx + c,$$

are obviously parallel. They have the same direction, but a different position.

31. *To find the equation of a line passing through A (x_1, y_1) and making an angle β with the x-axis, the axes of coordinates being rectangular.*

Let (x, y) be the coordinates of any point P on the line. Let r be the algebraical distance of the point from (x_1, y_1). Let r be considered positive if AP is in the first or second quadrants formed by axes through A parallel to the original axes, otherwise let r be negative.

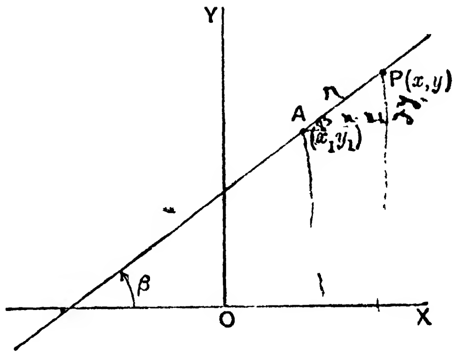
With this convention as to sign it will be seen that

$$x - x_1 = r\cos\beta,$$
$$y - y_1 = r\sin\beta.$$

So that we may express the line by the equations

$$\frac{x-x_1}{\cos\beta}=\frac{y-y_1}{\sin\beta}=r \quad \text{.....................(E)},$$

in which β is measured from 0 to π.

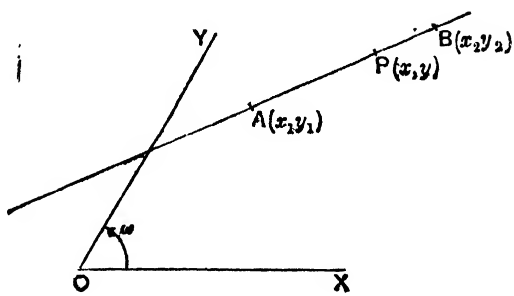


This is a very useful form and we shall have occasion to use it frequently in later chapters. It will be seen that by equations (E) the coordinates of any point on the line are at once expressed when we know its algebraical distance from the given point (x_1, y_1) for we have

$$x=x_1+r\cos\beta, \quad y=y_1+r\sin\beta.$$

32. It may be observed that if the axes of coordinates be oblique a straight line through (x_1, y_1) can still be expressed in the form

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=r,$$



where l and m, though not now the cosine and sine of the angle which the line makes with the x-axis, are constants depending only on the direction of the line.

For let B be a point on the line and let its coordinates be (x_2, y_2) and let the algebraical ratio of $AP:PB$ be $p:q$, P being any point on the line.

Then (x, y) being the coordinates of P, we have (§ 9)

$$x = \frac{px_2 + qx_1}{p+q}, \qquad y = \frac{py_2 + qy_1}{p+q};$$

$$\therefore \quad x - x_1 = \frac{p(x_2 - x_1)}{p+q}, \quad y - y_1 = \frac{p(y_2 - y_1)}{p+q};$$

$$\therefore \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{p}{p+q} = \frac{AP}{AB}.$$

This we may write

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r,$$

where
$$l = \frac{x_2 - x_1}{r_{12}}, \quad m = \frac{y_2 - y_1}{r_{12}},$$

r_{12} being the algebraical distance of B from A, as r is of P from A.

It is clear then that l and m depend only on the direction of the line and that we have

$$l^2 + m^2 + 2lm\cos\omega$$
$$= \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega}{r_{12}^2} = 1.$$

Use of the form $x\cos\alpha + y\sin\alpha = p$.

33. *To find the length of the perpendicular from the origin on the line*

$$Ax + By + C = 0.$$

We may without loss of generality take B to be positive in this equation, for if it were not positive we should have only to change the signs all through.

Our equation may be written

$$kA \,.\, x + kB \,.\, y + kC = 0,$$

where k is any constant other than zero.

Choose k so that $\quad (kA)^2 + (kB)^2 = 1.$

This is satisfied by $k = \dfrac{1}{\sqrt{A^2+B^2}}$, and we will take the positive sign with the radical.

Our equation is now

$$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y + \frac{C}{\sqrt{A^2+B^2}} = 0,$$

in which the coefficient of y is still positive.

Hence our equation is of the form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

where $\quad \cos \alpha = \dfrac{A}{\sqrt{A^2+B^2}}, \quad \sin \alpha = \dfrac{B}{\sqrt{A^2+B^2}},$

$$p = -\frac{C}{\sqrt{A^2+B^2}}.$$

Since $\sin \alpha$ is positive, α may represent an angle between 0 and π.

Thus we have reduced our equation to the form (C) and our convention of § 26 is applicable.

Thus perpendicular $\; p = -\dfrac{C}{\sqrt{A^2+B^2}}.$

If p is positive (that is C negative) the perpendicular between the origin and the line falls in one of the first two quadrants.

If p is negative (that is C positive) the perpendicular between the origin and the line falls in the third or the fourth quadrant.

Examples. 1. Write the equation $3x+4y-28=0$ in the form

$$x\cos\alpha+y\sin\alpha=p$$

and find the perpendicular from the origin upon the line.

2. Express $4x-3y=18$ in the form $x\cos\alpha+y\sin\alpha-p=0$ (α between 0 and π) and find the perpendicular from the origin upon it.

Verify that the sign of your perpendicular is correct by drawing the straight line in a figure.

3. Find the perpendicular from the origin on the lines

$$(1)\quad 5x+6y=7,$$
$$(2)\quad 6x-7y-8=0,$$

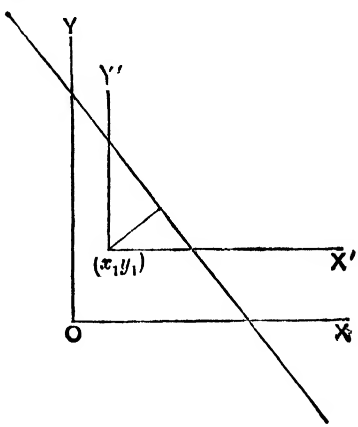
and state in which quadrant each falls.

34. *To find the length of the perpendicular from the point* (x_1, y_1) *on the line*

$$Ax+By+C=0.$$

The method we shall adopt is as follows. We shall transfer the origin to the point (x_1, y_1) and obtain the new equation of the line. We then use the method of § 33 to find the length of the perpendicular from the new origin. We shall as before take B to be positive.

Let (x, y) be the coordinates of any point P on the line referred to the original axes, let (X, Y) be the coordinates of



the same point P referred to axes through (x_1, y_1) parallel to the original axes

$$\therefore\ x=X+x_1,\qquad y=Y+y_1.$$

But $$Ax+By+C=0,$$

$$\therefore\ A(X+x_1)+B(Y+y_1)+C=0,$$

$$\therefore\ AX+BY+(Ax_1+By_1+C)=0,$$

$$\therefore\ \frac{A}{\sqrt{A^2+B^2}}X+\frac{B}{\sqrt{A^2+B^2}}Y+\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}=0;$$

this is the new equation of the line in the form

$$X\cos\alpha+Y\sin\alpha-p=0.$$

Hence $$p=-\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

This then is the perpendicular from the point (x_1, y_1) on the line

$$Ax+By+C=0.$$

35. If $-\dfrac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$ is positive, that is if Ax_1+By_1+C is negative, the perpendicular on the line falls in one of the first two quadrants formed by the new axes through (x_1, y_1).

If $-\dfrac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$ is negative, that is if Ax_1+By_1+C is positive, the perpendicular on the line falls in the third or fourth quadrant formed by the new axes through (x_1, y_1).

We thus get the following result:

If (x_1, y_1) be the coordinates of any point in the plane, and $Ax+By+C=0$ be the equation of a line (B positive) then Ax_1+By_1+C is positive if the point (x_1, y_1) be above the line (that is if the perpendicular from (x_1, y_1) on the line falls in the third or fourth quadrant formed by the axes through (x_1, y_1) parallel to the original axes), negative if (x_1, y_1) be below the line and zero if (x_1, y_1) be on the line.

Thus we see that any straight line $Ax+By+C=0$ divides the plane into two parts, such that the coordinates of all points on one side of the line substituted for x and y in $Ax+By+C$ make this expression positive, the coordinates of all points on

the other side of the line make the expression negative. Points on the line itself make the expression zero.

The relation $$Ax + By + C > 0$$
would be satisfied by all points above the line $Ax + By + C = 0$, always supposing B is positive. We might then speak of this side of the line as the positive side of the line. The other side we can call the negative side of the line. Points on the negative side make $Ax + By + C$ negative, that is satisfy
$$Ax + By + C < 0.$$

We see then that we can express analytically all points on one side of the line $Ax + By + C = 0$ by the inequality
$$Ax + By + C > 0$$
and all points on the other side of the line by the inequality
$$Ax + By + C < 0.$$

We have only shewn that those inequalities hold when the axes are rectangular but it will be seen later that they are applicable for oblique axes also.

Examples. 1. Shew that the perpendicular from (2, 3) on the line $4x+7y-18=0$ is of length $\frac{11}{\sqrt{65}}$ and that the point is above the line.

2. Find the perpendicular from (1, -3) on the same line and shew that the point is below the line.

3. Find the perpendiculars from (4, 3) and (-2, -1) on the line $5x+3y+8=0$. Determine on which side of the line each point lies, and verify by means of a figure.

4. Shew that the point (5, 6) is above the line $2x-4y+7=0$ and determine its perpendicular distance from the line.

5. Shew that the points (2, 3) and (1, 2) are on opposite sides of the line $5x+7y-20=0$.

Use of the form $y = mx + b$.

36. It has now been seen how the form
$$x \cos \alpha + y \sin \alpha - p = 0$$
for the equation of a line enables us to obtain the perpendicular

distance of any point (x_1, y_1) from any line $Ax + By + C = 0$, and to determine on which side of the line the point lies. We go on now to shew the use of the form $y = mx + b$.

The condition that the lines $y = mx + b$, $y = m'x + b'$ should be parallel is $m = m'$, whether the axes be rectangular or oblique.

For if β be the inclination of the lines to the axis of x $m = \tan\beta = m'$ if the axes be rectangular (§ 28); and

$$m = \frac{\sin\beta}{\sin(\omega - \beta)} = m'$$

if the axes be inclined at angle ω (§ 29).

It is clear then that the condition that the lines

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0,$$

should be parallel is $$-\frac{A}{B} = -\frac{A'}{B'},$$

for the first line is $y = -\frac{A}{B}x - \frac{C}{B}$ and its 'm' is $-\frac{A}{B}$ and the second line is $y = -\frac{A'}{B'}x - \frac{C'}{B'}$ and its 'm' is $-\frac{A'}{B'}$.

Thus if the two lines are parallel

$$\frac{A'}{A} = \frac{B'}{B} = k \text{ (say)}.$$

$$\therefore \quad A' = kA, \qquad B' = kB.$$

Thus the line $A'x + B'y + C' = 0$, parallel to $Ax + By + C = 0$, can have its equation written

$$k(Ax + By) + C' = 0,$$

or dividing out by k

$$Ax + By + D = 0.$$

This then is the general form of lines parallel to

$$Ax + By + C = 0.$$

Examples. [The axes are not necessarily rectangular here.]

1. Find the equation of the line through (2, 3) and parallel to the line $5x+8y=9$.

[The equation of the line required is of the form

$$5x+8y=k,$$

in which k must be so determined that the line passes through (2, 3).

The condition for this is

$$10+24=k,$$

$$\therefore\ k=34.$$

Therefore the line required is $5x+8y=34$.]

2. Find the equation of the line through (1, 3) parallel to $3x-4y=8$.

3. Find the equation of a line parallel to $2x+14y=7$ and making an intercept of 3 on the axis of y.

4. Find the equation of a line parallel to $3x-8y=21$ and making an intercept of -7 on the x-axis.

37. *The condition that the lines $y=mx+b$, $y=m'x+b'$ should be perpendicular is $mm'+1=0$, the axes being rectangular.*

For if β, β' be the inclinations of the lines to the axis of x

$$\beta' \sim \beta = \frac{\pi}{2}.$$

$$\therefore\ \beta'=\beta+\frac{\pi}{2},\ \text{or}\ \beta=\beta'+\frac{\pi}{2}.$$

In either case $\tan\beta = -\cot\beta'$,

$$\therefore\ \tan\beta\tan\beta' = -1,$$

$$\therefore\ mm' = -1.$$

This is a very important result.

It follows that the condition that the lines

$$Ax+By+C=0,$$

$$A'x+B'y+C'=0$$

should be perpendicular is

$$\left(-\frac{A}{B}\right)\left(-\frac{A'}{B'}\right)=-1.$$

$$\therefore\ AA'+BB'=0.$$

38. It follows from the last article that the general form for the equation of lines perpendicular to $Ax+By+C=0$ is (when the axes are rectangular) either

$$Bx - Ay = k,$$

or

$$\frac{x}{A} - \frac{y}{B} = l.$$

The k or l must be determined by some other datum respecting the line, as, for example, that it is to go through a given point.

Examples. 1. Shew that the lines $5x+6y=18$, $18x-15y=31$ are perpendicular.

2. Find the equation of the line through (2, 5) perpendicular to the line $2x+5y+31=0$.

[The equation of a perpendicular line is

$$5x-2y=k.$$

Choose k so that (2, 5) lies on this line,

$$\therefore\ 10-10=k,$$

$$\therefore\ k=0.$$

Thus the equation required is $5x-2y=0$.]

3. Find the equation of the line through (4, 5) perpendicular to the line $x-15y=20$.

39. General equation of lines through a given point.

The equation of all lines through the point (x_1, y_1) are included in

$$Ax+By=Ax_1+By_1.$$

For the general equation of a line is

$$Ax+By=C.$$

If this passes through (x_1, y_1)

$$Ax_1+By_1=C.$$

This gives C in terms of A and B and the coordinates of the given point.

Hence the general equation of all lines through (x_1, y_1) is

$$Ax+By=Ax_1+By_1,$$

or

$$A(x-x_1)+B(y-y_1)=0.$$

40. From the preceding article we see that the equation of the line through (x_1, y_1) parallel to $Ax + By + C = 0$ is

$$Ax + By = Ax_1 + By_1,$$

and the equation of the line through (x_1, y_1) perpendicular to the above is

$$Bx - Ay = Bx_1 - Ay_1,$$

or
$$\frac{x}{A} - \frac{y}{B} = \frac{x_1}{A} - \frac{y_1}{B}.$$

It is thus easy to write down quickly the equation of a line through a given point parallel to or perpendicular to a given line. For the terms in x and y are easily expressed in each case; these form the left side of the equation and the right-hand side is obtained by substituting the *special* coordinates (x_1, y_1) for the general ones on the left.

Thus the equation of a line through (3, 7) parallel to $2x + 5y = 9$ is

$$2x + 5y = 2 \times 3 + 5 \times 7,$$

that is
$$2x + 5y = 41.$$

And the equation of the line through (3, 7) perpendicular to $2x + 5y = 9$ is

$$5x - 2y = 5 \times 3 - 2 \times 7,$$

that is
$$5x - 2y = 1.$$

Examples. 1. Write down the equation of the lines through $(2, -3)$ respectively parallel and perpendicular to $4x - 7y = 1$.

2. Write down the equation of a line through $(4, -1)$ perpendicular to $2y - 3x = 7$.

41. Intersection of two lines. If we wish to find the point of intersection of two lines whose equations are

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

we have only to solve these equations as simultaneous. For the point where they intersect must be such that its coordinates satisfy *both* equations, and the coordinates of no other point will satisfy them both simultaneously.

42. Lines through the point of intersection of given lines.

If $$Ax+By+C=0 \text{.......................}(1),$$
and $$A'x+B'y+C'=0 \text{.......................}(2),$$
be the equations of two lines, then, *whatever constant k be*, the equation
$$Ax+By+C+k(A'x+B'y+C')=0 \text{ ....}(3),$$
will represent a straight line, and for different values of k we shall get different straight lines.

Further *the straight lines included in* (3) *all pass through the point of intersection of* (1) *and* (2) for the values of x and y which satisfy (1) and (2) simultaneously satisfy (3) also.

Hence the equations of lines through the point of intersection of (1) and (2) can be got from the form (3).

Examples. 1. Find the point of intersection of the lines $2x+7y=25$ and $7x-2y=8$.

2. Find the equation of the line through the point (3, 4) and the point of intersection of the lines $5x-y=9$, $x+6y=8$.

[*First method.*

Find the point of intersection of
$$5x-y=9,$$
$$x+6y=8.$$
We find $$x=2, \quad y=1.$$
The equation of the line joining this point (2, 1) to (3, 4) is
$$\frac{x-2}{2-3}=\frac{y-1}{1-4},$$
that is $$3x-y=5.$$

Second method.

The general equation of lines through the point of intersection of the given lines is
$$5x-y-9+k(x+6y-8)=0.$$
Choose k so that this passes through (3, 4),
$$\therefore\ 15-4-9+k(3+24-8)=0,$$
$$\therefore\ k=-\tfrac{2}{19}.$$

The equation of the required line is therefore

$$5x-y-9-\tfrac{2}{19}(x+6y-8)=0,$$

which reduces to $$93x-31y=155,$$

that is $$3x-y=5.]$$

3. Find the equation of the straight line through the origin and the intersection of the lines $5x+6y=23$, $3x-4y+9=0$.

4. Find the equation of the line through (2, 5) and the point of intersection of the lines $5x+6y=20$, $4x+9=17y$.

43. Condition for concurrence of three lines. The condition that the three lines

$$A_1x+B_1y+C_1=0 \quad \dots\dots\dots\dots(1),$$

$$A_2x+B_2y+C_2=0 \quad \dots\dots\dots\dots(2),$$

$$A_3x+B_3y+C_3=0 \quad \dots\dots\dots\dots(3),$$

should be concurrent or meet in a point is that the values of x and y which satisfy (1) and (2) simultaneously should satisfy also (3). The condition that the three equations should all be satisfied by the same values of x and y can be expressed at once in the form of a determinant:

$$\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix}=0,$$

which is the same as

$$A_1(B_2C_3-B_3C_2)+A_2(B_3C_1-B_1C_3)+A_3(B_1C_2-B_2C_1)=0.$$

44. Test for concurrence in special cases. If three straight lines have their equations as in § 43, and if three constants k, l, m can be found so that

$$k(A_1x+B_1y+C_1)+l(A_2x+B_2y+C_2)$$
$$+m(A_3x+B_3y+C_3)=0 \quad \dots\dots(4),$$

identically for *all* values of x and y, then the three lines are concurrent.

For let (x_1, y_1) be the point of intersection of (1) and (2),

$$\therefore\ A_1x_1+B_1y_1+C_1=0,$$

$$A_2x_1+B_2y_1+C_2=0.$$

Hence as (4) holds for *all* values of x and y it holds for $x = x_1$, $y = y_1$,

$$\therefore\ A_3 x_1 + B_3 y_1 + C_3 = 0.$$

Thus (x_1, y_1) lies also on (3).

Examples. 1. Find for what value of a the three lines

$$3x + y + 2 = 0,$$
$$2x - y + 3 = 0,$$
$$x + ay - 3 = 0,$$

will meet in a point.

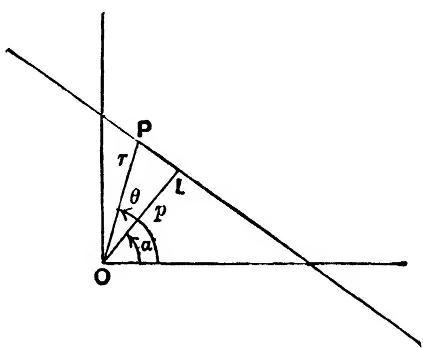
2. Prove that the lines $4x + 7y - 9 = 0$, $5x - 8y + 15 = 0$, $9x - y + 6 = 0$ are concurrent.

[Use § 44.]

3. The condition that the lines $a_1 x + b_1 y + 1 = 0$, $a_2 x + b_2 y + 1 = 0$, $a_3 x + b_3 y + 1 = 0$ should be concurrent is that the points (a_1, b_1) (a_2, b_2) (a_3, b_3) should be collinear.

45. Polar equation of a straight line. From the equation

$$x \cos \alpha + y \sin \alpha = p$$



we can, by writing $x = r \cos \theta$, $y = r \sin \theta$, where r and θ are the polar coordinates of any point on the line, obtain

$$r \cos (\theta - \alpha) = p \quad \text{.....................(F)},$$

which is the standard form of the polar equation of a line.

The same equation can be obtained from the accompanying figure in which OL being the perpendicular on the line and P any point on the line

$$OL = OP \cos LOP.$$

46. The general equation of a straight line is

$$Ax + By + C = 0.$$

Taking the axes as rectangular and writing $x = r\cos\theta$, $y = r\sin\theta$ we obtain the general polar equation of a line in the form

$$A\cos\theta + B\sin\theta = -\frac{C}{r}.$$

If then C be not zero, that is if the line does not pass through the origin, we may multiply by $-\frac{k}{C}$ and get the equation in the form

$$\frac{k}{r} = -\frac{kA}{C}\cos\theta - \frac{kB}{C}\sin\theta$$

$$= A'\cos\theta + B'\sin\theta.$$

Thus the general polar equation of a line is of the form

$$A\cos\theta + B\sin\theta = \frac{k}{r} \quad \text{..................(G)},$$

where k is any constant we are pleased to take, except in the case where the line goes through the origin. The equation of a line through the origin is

$$\theta = \text{constant}.$$

47. Parallel lines. It is clear that the lines

$$p = r\cos(\theta - \alpha),$$

$$p' = r\cos(\theta - \alpha)$$

are parallel.

As also are $\frac{k}{r} = A\cos\theta + B\sin\theta,$

$$\frac{k'}{r} = A\cos\theta + B\sin\theta.$$

That is to say the general equation of lines parallel to

$$\frac{k}{r} = A\cos\theta + B\sin\theta,$$

is obtained by changing the constant k.

48. Perpendicular lines. The lines

$$p = r\cos(\theta - \alpha),$$
$$p' = r\cos(\theta - \alpha')$$

will be perpendicular if $\alpha' - \alpha = \frac{\pi}{2}$.

Further the lines

$$\frac{k}{r} = A\cos\theta + B\sin\theta,$$

$$\frac{k'}{r} = B\cos\theta - A\sin\theta$$

are perpendicular (§ 38).

Thus the general equation of lines perpendicular to

$$\frac{k}{r} = A\cos\theta + B\sin\theta$$

can be obtained by writing $\frac{\pi}{2} + \theta$ for θ and changing the constant k.

Therefore of course the general equation of lines perpendicular to

$$p = r\cos(\theta - \alpha)$$

can be obtained by writing $\theta + \frac{\pi}{2}$ for θ and changing the constant p.

49. Area of triangle formed by three given lines.

To find the area of the triangle, the equations of whose sides are

$$a_1x + b_1y + c_1 = 0 \qquad \text{.......................(1)},$$
$$a_2x + b_2y + c_2 = 0 \qquad \text{.......................(2)},$$
$$a_3x + b_3y + c_3 = 0 \qquad \text{.......................(3)}.$$

We might find the area required by obtaining the coordinates of the three vertices, which we should get by solving these equations in pairs. The following method is however better, as by it we obtain an easily remembered formula. Let (x_1, y_1) (x_2, y_2) (x_3, y_3) be the coordinates of the vertices, (x_1, y_1)

being the intersection of (2) and (3), (x_2, y_2) of (3) and (1), and (x_3, y_3) of (1) and (2).

Then the area of the triangle is

$$\frac{1}{2}\sin\omega \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}$$

and this

$$= \frac{\frac{1}{2}\sin\omega \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}}$$

$$= \frac{\frac{1}{2}\sin\omega \begin{vmatrix} a_1x_1+b_1y_1+c_1, & a_2x_1+b_2y_1+c_2, & a_3x_1+b_3y_1+c_3 \\ a_1x_2+b_1y_2+c_1, & a_2x_2+b_2y_2+c_2, & a_3x_2+b_3y_2+c_3 \\ a_1x_3+b_1y_3+c_1, & a_2x_3+b_2y_3+c_2, & a_3x_3+b_3y_3+c_3 \end{vmatrix}}{\Delta}$$

where Δ stands for the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Thus, as all the constituents of the determinant in the numerator are zero except the three diagonal ones read downwards from left to right, we see that the area is

$$\frac{\frac{1}{2}\sin\omega\,(a_1x_1+b_1y_1+c_1)(a_2x_2+b_2y_2+c_2)(a_3x_3+b_3y_3+c_3)}{\Delta}.$$

Now let $\qquad a_1x_1+b_1y_1+c_1=k_1.$

We thus have

$$a_1x_1+b_1y_1+c_1-k_1=0,$$
$$a_2x_1+b_2y_1+c_2=0,$$
$$a_3x_1+b_3y_1+c_3=0.$$

Hence
$$\begin{vmatrix} a_1, & b_1, & c_1 - k_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0.$$

That is
$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} a_1, & b_1, & -k_1 \\ a_2, & b_2, & 0 \\ a_3, & b_3, & 0 \end{vmatrix} = 0,$$

$$\therefore \quad \Delta - k_1 \begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix} = 0.$$

That is
$$k_1 = \frac{\Delta}{C_1},$$
where C_1 is the minor of c_1 in Δ.

In the same way if we write
$$a_2x_2 + b_2y_2 + c_2 = k_2,$$
and
$$a_3x_3 + b_3y_3 + c_3 = k_3,$$
we shall get
$$k_2 = -\frac{\Delta}{C_2},$$
and
$$k_3 = \frac{\Delta}{C_3},$$
where C_2 and C_3 are the actual minors of c_2 and c_3 in Δ.

Thus the area is
$$-\frac{\frac{1}{2}\sin\omega\,\Delta^2}{C_1C_2C_3}.$$

If C_1', C_2', C_3' be the prepared minors of c_1, c_2, c_3, that is the actual minors taken with their proper signs, then
$$C_1' = C_1, \quad C_2' = -C_2, \quad C_3' = C_3.$$

Therefore the area is
$$\frac{\frac{1}{2}\sin\omega\,\Delta^2}{C_1'C_2'C_3'}.$$

Examples. 1. Find the area of the triangle formed by the lines
$$2x+y-3=0,$$
$$3x+2y-1=0,$$
$$2x+3y+4=0,$$
the axes being rectangular.

[The numerical value of the area is

$$\frac{\tfrac{1}{2}\begin{vmatrix} 2, & 1, & -3 \\ 3, & 2, & -1 \\ 2, & 3, & 4 \end{vmatrix}^2}{\begin{vmatrix} 3, & 2 \\ 2, & 3 \end{vmatrix}\begin{vmatrix} 2, & 1 \\ 2, & 3 \end{vmatrix}\begin{vmatrix} 2, & 1 \\ 3, & 2 \end{vmatrix}}$$

$$=\frac{\tfrac{1}{2}(16-2-27+12+6-12)^2}{5\times 4\times 1}=\frac{\tfrac{1}{2}\times 49}{20}=\frac{49}{40}.\Big]$$

2. Find the area of the triangle the equations of whose sides are

$$x+y+2=0,$$
$$2x-y-3=0,$$
$$3x+2y-5=0.$$

EXAMPLES.

1. The condition that the lines $y=mx+b$, $y=m'x+b'$ should be at right angles, when the axes are inclined at an angle ω is

$$mm'+(m+m')\cos\omega+1=0.$$

[Let β, β' be the inclinations of the lines to the axis of x,

$$\therefore\ \beta'\sim\beta=\frac{\pi}{2},$$

$$\therefore\ \tan\beta\tan\beta'-1=0.$$

But $$m=\frac{\sin\beta}{\sin(\omega-\beta)} \text{ and } m'=\frac{\sin\beta'}{\sin(\omega-\beta')}.\Big]$$

2. The condition that the lines

$$Ax+By+C=0,\quad A'x+B'y+C'=0$$

should be at right angles is

$$AA'+BB'-(AB'+A'B)\cos\omega=0.$$

3. Shew that the line joining the points (x_1, y_1) and (x_2, y_2) is cut by the line $Ax+By+C=0$ in the ratio

$$-\frac{Ax_1+By_1+C}{Ax_2+By_2+C}.$$

4. Obtain the polar equation of the line passing through the points whose polar coordinates are (r_1, θ_1), (r_2, θ_2) in the form

$$\frac{\sin(\theta_1-\theta_2)}{r}=\frac{\sin(\theta-\theta_2)}{r_1}-\frac{\sin(\theta-\theta_1)}{r_2}.$$

5. The diagonals of the parallelogram formed by the lines

$$ax + by + c = 0, \quad a'x + b'y + c = 0,$$
$$ax + by + c' = 0, \quad a'x + b'y + c' = 0,$$

will be at right angles if $a^2 + b^2 = a'^2 + b'^2$, the axes being rectangular.

6. The Cartesian equations of the sides BC, CA, AB of a triangle are

$$u_1 \equiv a_1x + b_1y + c_1 = 0, \quad u_2 \equiv a_2x + b_2y + c_2 = 0, \quad u_3 \equiv a_3x + b_3y + c_3 = 0,$$

and a line is drawn through A parallel to BC, prove that its equation is

$$\frac{u_3}{a_3b_1 - a_1b_3} + \frac{u_2}{a_1b_2 - a_2b_1} = 0.$$

Shew also that the equation of the line through A bisecting the side BC is

$$\frac{u_3}{a_3b_1 - a_1b_3} - \frac{u_2}{a_1b_2 - a_2b_1} = 0.$$

7 Find the coordinates of the centre of the circle inscribed in the triangle the equations of whose sides referred to rectangular axes are

$$x - y + 1 = 0, \quad x + y - 7 = 0, \quad x - 3y + 5 = 0.$$

Find also the three ecentres and discriminate between them.

8. Given that the origin lies at the escribed centre opposite $a_3x + b_3y + c_3 = 0$ of the triangle formed by this line and the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, shew that the centres of the circles touching the sides of the triangle are given by

$$\frac{a_1x + b_1y + c_1}{c_1} = \pm \frac{a_2x + b_2y + c_2}{c_2} = \pm \frac{a_3x + b_3y + c_3}{c_3},$$

and distinguish the cases.

CHAPTER IV.

PAIRS OF STRAIGHT LINES.

50. Angle between two lines.

To find the angle between the lines

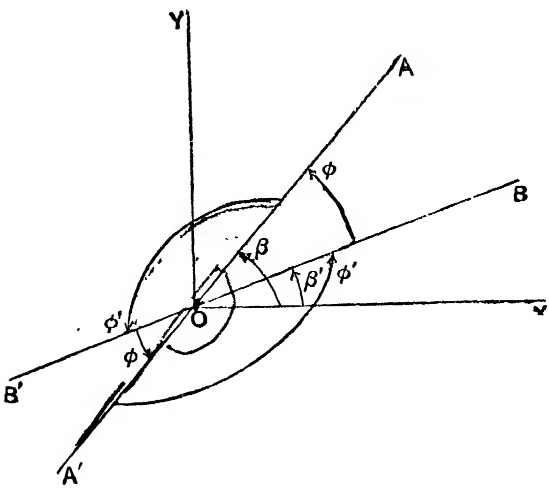
$$y = mx + b$$
$$y = m'x + b'$$

the axes being rectangular.

The angle between these lines is the same as the angle between the lines

$$y = mx \qquad\qquad (1),$$
$$y = m'x \qquad\qquad (2),$$

these being through the origin parallel to the above.



Let β and β' be the angles which the lines make with the x axis so that

$$\tan\beta = m, \quad \tan\beta' = m'.$$

Let $\beta > \beta'$, and ϕ the angle between the lines as shewn in the figure;

$$\therefore \quad \phi = \beta - \beta',$$

$$\therefore \quad \tan\phi = \frac{\tan\beta - \tan\beta'}{1 + \tan\beta\tan\beta'} = \frac{m - m'}{1 + mm'},$$

$$\therefore \quad \phi = \tan^{-1}\frac{m - m'}{1 + mm'}.$$

This is the angle between the lines measured positively from

$$y = m'x \text{ to } y = mx.$$

Therefore $\tan^{-1}\dfrac{m' - m}{1 + mm'}$, is the angle between them measured from $y = mx$ to $y = m'x$.

Thus in the figure

$\angle BOA$ or $B'OA'$ $(= \phi)$ is given by $\tan\phi = \dfrac{m - m'}{1 + mm'}$,

$\angle AOB'$ or $A'OB$ $(= \phi')$ is given by $\tan\phi' = \dfrac{m' - m}{1 + mm'}$.

Cor. If the lines be parallel $m' = m$ for $\phi = 0$.

If the lines be perpendicular $mm' + 1 = 0$ for $\phi = \dfrac{\pi}{2}$,

$$\therefore \quad \tan\phi = \infty.$$

These results we have already obtained independently in the previous chapter.

51. The angle between the lines

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0,$$

is at once obtained by writing $m = -\dfrac{A}{B}$, $m' = -\dfrac{A'}{B'}$ in the formula of the last article which the student will find easy to remember.

Equation of a pair of lines.

52. The equation

$$(Ax + By + C)(A'x + B'y + C') = 0 \quad \text{.........(1)},$$

can only be satisfied by points whose coordinates satisfy

either $$Ax + By + C = 0,$$

or $$A'x + B'y + C' = 0.$$

Thus all points which satisfy (1) lie on one or other of two fixed lines. The equation (1) therefore represents a *pair of straight lines.*

If we were to multiply the two factors together we should get an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{............(2)}$$

in which

$$a = AA' \qquad 2f = BC' + B'C$$

$$b = BB' \qquad 2g = CA' + C'A$$

$$c = CC' \qquad 2h = AB' + A'B.$$

An equation of the form (2) then will always represent two straight lines if it be the product of two 'linear' factors of the form

$$Ax + By + C, \quad A'x + B'y + C'.$$

There must be a relation between the coefficients a, b, c, f, g, h if this is to be the case. Without such relation the equation (2) will not represent a pair of lines.

We may here remark that equation (2) is known as *the general equation of the second degree.* It contains all possible terms of the second, first, and no degree in x and y.

53. We can sometimes tell at sight when an equation of the second degree represents a pair of lines. For example consider the equation

$$xy - 3x - 4y + 12 = 0.$$

This is obviously

$$(x - 4)(y - 3) = 0.$$

Thus the equation represents the two lines

$$x = 4,$$
$$y = 3,$$

which are parallel to the x and y axes respectively.

54. Equation of pair of lines through the origin. We are going on in the next article to obtain the condition that the general equation of the second degree should represent two straight lines. But let it be noticed first that an equation of the form

$$ax^2 + 2hxy + by^2 = 0$$

always *represents a pair of straight lines both passing through the origin.*

For $$ax^2 + 2hxy + by^2 \equiv a(x - \alpha y)(x - \beta y),$$

where α and β are the roots of the equation

$$az^2 + 2hz + b = 0 \text{ in } z.$$

Hence the equation

$$ax^2 + 2hxy + by^2 = 0$$

represents the lines $$x = \alpha y$$
$$x = \beta y$$

which pass through the origin.

55. Condition that the general equation of the second degree should represent two straight lines.

Suppose that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)}$$

represents two straight lines.

Let (x_1, y_1) be the point of intersection of the lines.

Transfer the origin to (x_1, y_1) keeping the *direction* of the axes unchanged.

Let (X, Y) be the coordinates of any point on the locus (1) referred to the new axes, (x, y) of the same point referred to the original axes,

$$\therefore\ x = X + x_1, \quad y = Y + y_1.$$

But x, y satisfy (1),

$$\therefore\ a(X+x_1)^2+2h(X+x_1)(Y+y_1)+b(Y+y_1)^2 \\ +2g(X+x_1)+2f(Y+y_1)+c=0.$$

But both of the lines now pass through the origin; therefore the terms of first order in X, Y and the constant term must disappear and leave only

$$aX^2+2hXY+bY^2=0.$$

That is, we have

$$ax_1+hy_1+g=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$hx_1+by_1+f=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0\ldots\ldots\ldots(4).$$

These are equivalent to (2) and (3) and another equation formed from

$$(4)-(2)\times x_1-(3)\times y_1,$$

that is $$gx_1+fy_1+c=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Eliminate x_1 and y_1 from (2), (3) and (5) and thus get

$$abc+2fgh-af^2-bg^2-ch^2=0,$$

or, in determinant form,

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=0.$$

COR. We see from the above that if the general equation of the second degree represents two straight lines, the point where they intersect is given by

$$\left.\begin{array}{l} ax_1+hy_1+g=0 \\ hx_1+by_1+f=0 \end{array}\right\}$$

i.e. $$x_1=\frac{hf-bg}{ab-h^2}, \qquad y_1=\frac{gh-af}{ab-h^2}.$$

From which we see that if the general equation of the second degree represents a pair of lines, the lines will be parallel if

$$ab=h^2,$$

for then their point of intersection is at infinity.

56. We see that the *necessary* condition that the general equation of the second degree should represent two lines is

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

We can see further that this condition is *sufficient*. For if it hold, it will be possible to find x_1 and y_1 to satisfy

$$ax_1 + hy_1 + g = 0 \quad \text{..................... (1)},$$

$$hx_1 + by_1 + f = 0 \quad \text{..................... (2)},$$

$$gx_1 + fy_1 + c = 0 \quad \text{..................... (3)},$$

since the values of x_1, y_1 given by the first two equations satisfy the third.

$$(1) \times x_1 + (2) \times y_1 + (3)$$

gives $$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Now transfer the origin to (x_1, y_1) as given by (1) and (2) and the equation is seen to reduce to

$$aX^2 + 2hXY + bY^2 = 0$$

which is a pair of lines.

57. The condition that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{......... (1)},$$

should be a pair of lines can also be obtained as follows.

Treat the equation as a quadratic in x and we have

$$ax^2 + 2x(hy + g) + (by^2 + 2fy + c) = 0,$$

$$\therefore \; x = \frac{-(hy + g) \pm \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}}{a}.$$

Hence the left-hand side of (1) cannot be the product of two linear factors unless

$$(hy + g)^2 - a(by^2 + 2fy + c)$$

be a perfect square.

But this $= (h^2 - ab)y^2 + 2y(gh - af) + g^2 - ac.$

If this be a perfect square,

$$(gh-af)^2=(h^2-ab)(g^2-ac),$$

$$\therefore\ a^2f^2+g^2h^2-2afgh=g^2h^2-abg^2-ach^2+a^2bc,$$

$$\therefore\ abc+2fgh-af^2-bg^2-ch^2=0.$$

Examples. 1. Shew that the following equations represent each a pair of straight lines and draw the lines in a figure:

(i) $xy=0$, (iii) $xy-5x+6y-30=0$,

(ii) $x^2-4y^2=0$, (iv) $x^2+2xy-x-4y-2=0$.

2. Find for what value of c the equation

$$2x^2+6xy+y^2+4x+2y+c=0,$$

represents two straight lines.

3. For what value of a will the equation

$$ax^2-3xy+2y^2-4x+y-4=0,$$

represent two straight lines?

58. *If the general equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0\text{...........}(1),$$

represents a pair of straight lines, then the equation

$$ax^2+2hxy+by^2=0\text{....................}(2),$$

represents a pair of straight lines parallel to them through the origin.

For if (1) be

$$(Ax+By+C)(A'x+B'y+C')=0,$$

(2) must be $(Ax+By)(A'x+B'y)=0.$

And the lines

$$Ax+By=0, \qquad A'x+B'y=0,$$

are lines through the origin respectively parallel to

$$Ax+By+C=0 \text{ and } A'x+B'y+C'=0.$$

59. Perpendicularity of two lines. *To find the condition that the lines*

$$ax^2+2hxy+by^2=0,$$

should be perpendicular, the axes being rectangular.

Let $ax^2+2hxy+by^2\equiv(Ax+By)(A'x+B'y).$

The pair of lines will be perpendicular if

$$AA' + BB' = 0.$$

But $\qquad a = AA'$ and $b = BB'$,

$\therefore$ the condition for perpendicularity is

$$a + b = 0.$$

This condition is easily seen to be sufficient as well as necessary.

It follows that if the general equation of the second degree represents two straight lines, these will be at right angles if

$$a + b = 0.$$

The student will observe that this is only true if the axes be at right angles. The condition when the axes are oblique is deferred for the present.

Bisectors of the angles between two lines.

60. *To find the equation of the bisectors of the angles between the lines*

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0,$$

the axes being rectangular.

The bisectors are the locus of points such that the magnitude of the perpendiculars from them on the two lines are equal.

Hence if (x, y) be any point on one of the bisectors

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}.$$

The two bisectors can then be expressed in one equation thus:

$$\left[\frac{Ax + By + C}{\sqrt{A^2 + B^2}} + \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}\right]\left[\frac{Ax + By + C}{\sqrt{A^2 + B^2}} - \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}\right] = 0,$$

that is

$$\frac{(Ax + By + C)^2}{A^2 + B^2} - \frac{(A'x + B'y + C')^2}{A'^2 + B'^2} = 0.$$

The student will see that these bisectors are at right angles by applying the test of § 59.

61. *To obtain the equation of the straight lines bisecting the angles between the lines*

$$ax^2 + 2hxy + by^2 = 0,$$

the axes being rectangular.

Let $\qquad ax^2 + 2hxy + by^2 \equiv a(x - \alpha y)(x - \beta y),$

$$\therefore \quad \alpha + \beta = -\frac{2h}{a} \quad \text{and} \quad \alpha\beta = \frac{b}{a}.$$

The equation of the bisectors is, as in § 60,

$$\frac{(x - \alpha y)^2}{1 + \alpha^2} - \frac{(x - \beta y)^2}{1 + \beta^2} = 0.$$

That is

$$(1 + \beta^2)(x^2 - 2\alpha xy + \alpha^2 y^2) - (1 + \alpha^2)(x^2 - 2\beta xy + \beta^2 y^2) = 0,$$

that is

$$x^2(\beta^2 - \alpha^2) + 2xy(\beta - \alpha)(1 - \alpha\beta) - (\beta^2 - \alpha^2)y^2 = 0,$$

that is $\qquad x^2 + 2\dfrac{1 - \alpha\beta}{\beta + \alpha}xy - y^2 = 0,$

that is $\qquad x^2 + 2\dfrac{1 - \dfrac{b}{a}}{-\dfrac{2h}{a}}xy - y^2 = 0,$

that is $\qquad \dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}.$

62. *To find the angle between the pair of lines*

$$ax^2 + 2hxy + by^2 = 0,$$

the axes being rectangular.

Let $\qquad ax^2 + 2hxy + by^2 \equiv b(y - mx)(y - m'x).$

$$\therefore \quad m + m' = -\frac{2h}{b}, \quad mm' = \frac{a}{b}.$$

Let ϕ be the angle between the lines,

$$\therefore \quad \tan^2\phi = \left(\frac{m - m'}{1 + mm'}\right)^2 \qquad (\S\ 50)$$

$$= \frac{(m + m')^2 - 4mm'}{(1 + mm')^2}$$

$$= \frac{\dfrac{4h^2}{b^2} - \dfrac{4a}{b}}{\left(1 + \dfrac{a}{b}\right)^2} = \frac{4(h^2 - ab)}{(a+b)^2},$$

$$\therefore \quad \tan\phi = \pm \frac{2\sqrt{h^2 - ab}}{(a+b)},$$

the + or − sign being taken according as ϕ is the acute or obtuse angle between the lines.

Implied in the formula for $\tan\phi$ is the condition for perpendicularity, viz.

$$a + b = 0.$$

63. We will conclude this chapter with an important proposition which will be useful in subsequent chapters.

We have already said that the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{............}(1),$$

only represents a pair of straight lines in special cases.

Generally it represents a curve, viz. a circle or other conic section as we shall prove later on.

Now suppose we have the equation of a line

$$lx + my = 1 \text{..........................}(2).$$

Form an equation by making (1) homogeneous of the second order in x and y by means of (2): viz. the equation

$$ax^2 + 2hxy + by^2 + 2(gx + fy)(lx + my) + c(lx + my)^2 = 0 \text{...}(3).$$

This we know represents a pair of straight lines through the origin. Moreover (3) is satisfied by points which satisfy (1) and (2) simultaneously.

Hence (3) represents a pair of straight lines through the origin and the points of intersection of the line (2) with the curve (1).

If the line were given as

$$lx + my = n,$$

instead of in the form (2), the equation of the pair of lines would be

$$(ax^2 + 2hxy + by^2)\,n^2 + 2\,(gx + fy)\,(lx + my)\,n + c\,(lx + my)^2 = 0.$$

EXAMPLES.

(The axes of coordinates are to be taken rectangular unless otherwise stated.)

1. Prove that the angle between the lines

$$y = mx + b, \quad y = m'x + b'$$

when the axes are inclined at an angle ω is

$$\tan^{-1} \frac{(m - m')\sin\omega}{1 + (m + m')\cos\omega + mm'}.$$

[If β be the angle which the first line makes with the x axis

$$m = \frac{\sin\beta}{\sin(\omega - \beta)} \qquad \text{whence } \tan\beta = \frac{m\sin\omega}{1 + m\cos\omega}.$$

Use method of § 50.]

2. Shew that the angle between the lines

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\tan^{-1} \frac{2\sqrt{h^2 - ab}\,\sin\omega}{a - 2h\cos\omega + b},$$

where ω is the angle between the axes.

[Use the method of § 62.]

3. The equation of the line through the origin making an angle ϕ with the line $y = mx + b$ is

$$y = \frac{m + \tan\phi}{1 - m\tan\phi}\,x.$$

4. Prove that the product of the perpendiculars from (x_1, y_1) on the lines given by

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}}.$$

5. The equation of the pair of lines through the origin perpendicular to the pair whose equation is

$$ax^2 + 2hxy + by^2 = 0,$$

is

$$bx^2 - 2hxy + ay^2 = 0.$$

6. The condition that the pair of lines

$$ax^2 + 2hxy + by^2 = 0 \text{ and } a'x^2 + 2h'xy + b'y^2 = 0$$

should have one line in common is

$$4(ah' - a'h)(hb' - h'b) = (ab' - a'b)^2.$$

7. If $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a pair of lines intersecting in (x_1, y_1) then the equation of the lines bisecting the angles between them will be

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h}.$$

8. If $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represent a pair of lines, the area of the triangle formed by their bisectors and the axis of x is

$$\frac{\sqrt{(a-b)^2 + 4h^2}}{2h} \cdot \frac{ca - g^2}{ab - h^2}.$$

9. Shew that the line $Ax + By + C = 0$, and the two lines represented by

$$(Ax + By)^2 - 3(Ay - Bx)^2 = 0$$

form the sides of an equilateral triangle.

10. Shew that all the pairs of lines

$$ax^2 + 2hxy + by^2 = \lambda(x^2 + y^2)$$

for different values of λ have the same bisectors.

11. Shew that the four lines given by the equations

$$(y - mx)^2 = c^2(1 + m^2)$$

$$(y - nx)^2 = c^2(1 + n^2)$$

form a rhombus.

12. The vertices of a triangle lie on the lines

$$y = x \tan \theta_1, \quad y = x \tan \theta_2, \quad y = x \tan \theta_3,$$

the circumcentre being at the origin; prove that the locus of the orthocentre is the line

$$x (\sin \theta_1 + \sin \theta_2 + \sin \theta_3) - y (\cos \theta_1 + \cos \theta_2 + \cos \theta_3) = 0.$$

13. The distance from the origin to the orthocentre of the triangle formed by the lines $\frac{x}{\alpha} + \frac{y}{\beta} = 1$

and $$ax^2 + 2hxy + by^2 = 0$$

is $$\frac{(a + b)\, \alpha\beta\, (\alpha^2 + \beta^2)^{\frac{1}{2}}}{a\alpha^2 - 2h\alpha\beta + b\beta^2}.$$

14. Prove that the equation

$$(a + 2h + b)\, x^2 - 2 (a - b)\, xy + (a - 2h + b)\, y^2 = 0$$

denotes a pair of straight lines each inclined at an angle of 45° to one or other of the lines given by

$$ax^2 + 2hxy + by^2 = 0.$$

15. Shew that the centroids of the triangles of which the three perpendiculars lie along the lines

$$y - m_1 x = 0, \quad y - m_2 x = 0, \quad y - m_3 x = 0$$

lie on $y (3 + m_2 m_3 + m_3 m_1 + m_1 m_2) = x (m_1 + m_2 + m_3 + 3 m_1 m_2 m_3)$.

16. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines

$$ax^2 + 2hxy + by^2 = 0.$$

Prove that the locus of its vertex is

$$(a + b)(x^2 + y^2) + 2h (fy + gx) + (a - b)(fx - gy) = 0.$$

17. Two equilateral triangles ABC, PQR have the same centre, the order of the letters for each triangle corresponding to circuits in the opposite sense. Prove that AP, BQ, CR are concurrent.

18. Shew that the equation

$$ax^2 + 2hxy + by^2 = \{(a+b)\sin^2\theta + (h^2 - ab)^{\frac{1}{2}}\sin 2\theta\}(x^2 + y^2)$$

represents two straight lines having the same bisectors as

$$ax^2 + 2hxy + by^2 = 0,$$

and making equal angles θ with them respectively.

19. Shew that the equation

$$(ab - h^2)(ax^2 + 2hxy + by^2 + 2gx + 2fy) + af^2 + bg^2 - 2fgh = 0$$

represents a pair of straight lines; and that these straight lines form a rhombus with the lines

$$ax^2 + 2hxy + by^2 = 0$$

provided that

$$(a-b)fg + h(f^2 - g^2) = 0.$$

20. On the sides of a triangle as diagonals, parallelograms are described having their sides parallel to two given straight lines; prove that the other diagonals of these parallelograms are concurrent

CHAPTER V.

THE CIRCLE.

64. Equation of a circle. The equation of a circle whose centre is at (h, k) and whose radius is a is clearly

$$(x-h)^2+(y-k)^2=a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

if the axes are rectangular; and

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=a^2 \quad\ldots(2)$$

if the axes are inclined at an angle ω.

This is so because by these equations is expressed the fact that the square of the distance of the point (x, y) from (h, k) is a^2.

65. The equation

$$x^2+y^2+2gx+2fy+c=0$$

will represent a circle if the axes are rectangular.

For this is equivalent to

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

which expresses the fact that the square of the distance of the point (x, y) from $(-g, -f)$ is constant and equal to g^2+f^2-c.

The above then is a circle having its centre at $(-g, -f)$ and its radius equal to $\sqrt{g^2+f^2-c}$.

We see then that the conditions that the general equation of the second degree, viz.

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

should represent a circle *when the axes are rectangular* are:

$$a=b,$$
$$h=0.$$

These however are not the conditions that the general equation should represent a circle when the axes are oblique. For the general equation of a circle when the axes are inclined at any angle ω is, as we see from (2) of § 64, of the form

$$x^2 + 2xy \cos \omega + y^2 + 2gx + 2fy + c = 0.$$

The conditions then for a circle would be

$$a = b$$

and $$h = a \cos \omega.$$

It is but rarely that we use oblique axes when we have to do with circles that we shall not trouble the student more with them in this chapter and shall confine ourselves throughout the chapter to the case of rectangular axes.

66. *To find the equation of the circle described on the line joining two given points A (x_1, y_1) and B (x_2, y_2) as diameter.*

The circle required is the locus of points P such that APB is a right angle.

Let (x, y) be the coordinates of a point P on the locus.

The 'm' of the line AP is $\dfrac{y - y_1}{x - x_1}$.

The 'm' of the line BP is $\dfrac{y - y_2}{x - x_2}$.

$\therefore$ as AP and BP are at right angles

$$\frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} + 1 = 0.$$

That is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

This then is the required equation.

Examples. 1. Find the centre and radius of each of the circles

(i) $x^2 + y^2 - 4x - 6y + 3 = 0$,

(ii) $x^2 + y^2 + 3x + 5y + 1 = 0$,

(iii) $2(x^2 + y^2) + 6x - 7y = 0$.

2. Find the equation of the circle passing through the origin P and cutting the axes of x and y at distances a and b from the origin.

3. Shew that the locus of a point such that the sum of the squares of its distances from two fixed points is constant is a circle.

4. Shew that the locus of a point such that the ratio of its distances from two given points is constant is a circle.

67. On the constants in the equation of a circle. We have seen that $-g$, and $-f$ are the x and y coordinates of the centre of the circle whose equation is

$$x^2+y^2+2gx+2fy+c=0.$$

We enquire naturally what is the geometrical meaning of the remaining constant c. An answer to this enquiry will be supplied when we have proved the proposition which is given in the next article.

68. **Prop.** *If from any point P (x_1, y_1) in the plane of the circle*

$$x^2+y^2+2gx+2fy+c=0$$

a line PQR be drawn to cut the circle in Q and R the product of the algebraical distances PQ and PR is independent of the direction of the line and equal to

$$x_1^2+y_1^2+2gx_1+2fy_1+c.$$

For let the line make an angle θ with the x-axis, then the line can be analytically expressed (§ 31) by the equations

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r,$$

where r is the algebraical distance of the point (x, y) on the line from (x_1, y_1).

We want now to find the distances from P of the points Q and R where the line cuts the circle.

So then write
$$x=x_1+r\cos\theta,$$
$$y=y_1+r\sin\theta,$$

and substitute into the equation of the circle.

We have

$$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2+2g(x_1+r\cos\theta)$$
$$+2f(y_1+r\sin\theta)+c=0.$$

This is a quadratic equation in r, the roots of which are the algebraical values of PQ and PR.

This equation when simplified becomes

$$r^2(\cos^2\theta+\sin^2\theta)+2r\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}$$
$$+x_1^2+y_1^2+2gx_1+2fy_1+c=0.$$

Thus as $\cos^2\theta+\sin^2\theta=1$, the product of the roots is

$$x_1^2+y_1^2+2gx_1+2fy_1+c.$$

Hence we see that the product of the algebraical distances PQ and PR is independent of the direction of the line PQR.

69. We have thus proved analytically what the student will have recognised to be the well-known geometrical property of the circle, that the rectangle of the segments of all chords drawn through a definite point are equal.

Now in the particular case where P is at the origin $x_1=0$, $y_1=0$ and the expression

$$x_1^2+y_1^2+2gx_1+2fy_1+c$$

reduces to c. Thus c is the constant rectangle of the segments of chords through the origin.

70. We see that if the point P be outside the circle PQ and PR will have the same sign, that is

$$x_1^2+y_1^2+2gx_1+2fy_1+c$$

will be positive; whereas if P be inside the circle PQ and PR will have opposite signs, and so

$$x_1^2+y_1^2+gx_1+2fy_1+c$$

will be negative.

Thus the circumference of the circle divides the plane into two regions. The coordinates of points outside the circle make $x^2+y^2+2gx+2fy+c$ positive, the coordinates of points inside the circle make this expression negative; while those of points on the circumference make the expression zero.

Thus $$x^2+y^2+2gx+2fy+c=0$$

represents the circumference of the circle,

$$x^2+y^2+2gx+2fy+c>0$$

is the analytical representation of all points outside the circle,

and $$x^2+y^2+2gx+2fy+c<0$$

is the representation of all points inside the circle.

The constant c will be positive if the origin be outside the circle, negative if it be inside the circle, zero if it be on the circumference.

71. We see from § 68 that every line through (x_1, y_1) will meet the circle in two points, since whatever be the value of θ we have a quadratic equation in r to determine the points common to the line and the circle.

The roots of this equation will not always be real, and in this case we say that the line meets the circle in two imaginary points.

If the roots of the quadratic in r be equal to one another we say that the line meets the circle in two coincident points and we call the line a tangent to the circle.

By a tangent to the circle then we shall understand a line in its plane which meets it in two coincident points.

On imaginary points.

72. The existence of imaginary points in Analytical Geometry is a matter of not a little importance. In elementary Pure Geometry we are accustomed to say that a line in the plane of a circle either meets the circle or it does not. In Analytical Geometry our principles compel us to say that *every* line in the plane of a circle meets the circle. There is no real contradiction between the two. For in Analytical Geometry we have the possibility of imaginary points, that

is to say, points whose coordinates are of the form $(\alpha+\beta\sqrt{-1}, \gamma+\delta\sqrt{-1})$, where α, β, γ, δ are real. These points have algebraical significance, but we cannot strictly represent them in a figure, and so they are non-existent in Pure Geometry until this takes up the ideas of Analytical Geometry, which indeed it has done to its great advantage.

73. In elementary Pure Geometry we say that two tangents can be drawn to a circle from a point outside it, one from a point on its circumference, and none from a point inside it. But in Analytical Geometry we must not make any such statement. We must, in accordance with our principles, say that from any point in the plane of a circle two tangents can be drawn to it; these tangents will be real if the point be outside the circle, they will be coincident if the point be on the circle, and they will be imaginary if the point be within the circle. Let us examine more closely why our principles require this of us.

We have seen in § 68 that the points in which a line through a given point (x_1, y_1) meets the circle are determined by the solution of a quadratic equation in r, viz.:

$$r^2+2r\{(x_1+g)\cos\theta+(y_1+f)\sin\theta\}$$
$$+x_1^2+y_1^2+2gx_1+2fy_1+c=0\text{............}(1).$$

Now if the line we have drawn be a tangent to the circle this equation must have equal roots, for a tangent meets the circle in two coincident points, that is, we must have

$$[(x_1+g)\cos\theta+(y_1+f)\sin\theta]^2=x_1^2+y_1^2+2gx_1+2fy_1+c,$$

from which we get

$$(x_1+g)\cos\theta+(y_1+f)\sin\theta=\pm\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}\text{...}(2).$$

By means of this and

$$\cos^2\theta+\sin^2\theta=1$$

$\cos\theta$ and $\sin\theta$ can be found.

Now if we take the positive sign with the radical, clearly we shall get two solutions, say

$$\cos\theta = A, \qquad \cos\theta = C,$$
$$\sin\theta = B, \qquad \sin\theta = D,$$

and if we take the negative sign we shall have

$$\cos\theta = -A, \qquad \cos\theta = -C,$$
$$\sin\theta = -B, \qquad \sin\theta = -D.$$

We get then two lines through (x_1, y_1) which are tangents, viz.

$$\frac{x-x_1}{A} = \frac{y-y_1}{B},$$

and $$\frac{x-x_1}{C} = \frac{y-y_1}{D}.$$

These lines are obviously imaginary if (x_1, y_1) be within the circle, for then the expression under the radical in equation (2) is negative. The lines are coincident if (x_1, y_1) be on the circumference of the circle, for then the expression under the radical vanishes, so that $C = -A$ and $D = -B$.

If then we define a tangent to a circle as a line meeting the circle in two coincident points and allow the algebra free play, we must admit that there are two tangents from every point in the plane of the circle which become coincident when the point lies on the circumference of the circle.

Further, it is clear that the lengths of the two tangents from any point are equal. For the length of a tangent from (x_1, y_1) is the value of r furnished by (1) *when it has equal roots.*

As the product of the roots is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$

we see that if t be the length of this tangent from (x_1, y_1)

$$t^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$

which is independent of θ.

Thus the square of each tangent from a point (x_1, y_1) to the circle is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$

whether (x_1, y_1) be without or within the circle.

We have made a somewhat long digression on imaginary points and lines because we think it important for the student to have clear ideas about them at this stage of the subject. Imaginary points and lines differ from real points and lines in that they have only algebraical significance, whereas real points and lines have geometrical significance too. Imaginary points and lines cannot be located in a figure as real ones can.

Examples. 1. The locus of points the tangents from which to two given circles bear to one another a constant ratio is a circle.

2. Find the length of the tangent from (2, 3) to the circle

$$2x^2 + 2y^2 - 3x + 4y = 0.$$

[The equation must first be divided by 2 to make the coefficients of x^2 and y^2 unity.]

74. Equation of the tangent at a point.

To find the equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

at the point (x_1, y_1) *on its circumference.*

The equations of a line through (x_1, y_1) making an angle θ with the x-axis are

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \text{.....................(1).}$$

So that a point situated on the line and distant r from (x_1, y_1) is given by

$$x = x_1 + r\cos\theta,$$

$$y = y_1 + r\sin\theta.$$

If this point be on the circle we must have

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta)$$
$$+ 2f(y_1 + r\sin\theta) + c = 0$$

which is a quadratic in r, viz.:

$$r^2 + 2r\{(x_1+g)\cos\theta + (y_1+f)\sin\theta\} + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Now if the line (1) be a tangent, both the values of r furnished by this equation must be zero, for the line is to meet the circle at two coincident points, viz. (x_1, y_1) itself.

We must therefore have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

which is satisfied, since (x_1, y_1) is by hypothesis on the circumference of the circle, and also

$$(x_1+g)\cos\theta + (y_1+f)\sin\theta = 0.$$

This gives us the direction of the line that it may be a tangent.

To get the actual equation of the tangent we must eliminate θ between this relation and equation (1).

The equation of the tangent is then

$$\frac{x-x_1}{\cos\theta}\times(x_1+g)\cos\theta = -\frac{y-y_1}{\sin\theta}\times(y_1+f)\sin\theta$$

that is $\qquad (x-x_1)(x_1+g) + (y-y_1)(y_1+f) = 0.$

By making use of the fact that (x_1, y_1) is on the circle we can reduce this down to a form which it is not difficult to remember.

The equation is

$$\begin{aligned} xx_1 + yy_1 + gx + fy &= x_1^2 + y_1^2 + gx_1 + fy_1 \\ &= -gx_1 - fy_1 - c; \end{aligned}$$

$$\therefore \; xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0.$$

This is the standard form of the equation of the tangent.

Circle referred to its centre.

75. In the special case where the origin is at the centre of the circle the equation of the circle assumes the simple form

$$x^2 + y^2 = a^2,$$

where a is the radius of the circle. If we wish to obtain the

geometrical properties of the circle by analytical methods we naturally take the equation of the circle in this simplest form. But we have sometimes to find properties of a circle in combination with other figures and then we may find it necessary to take the origin elsewhere than at the centre of the circle. The student then should accustom himself to the general form of the equation of the circle as we have up to this point been presenting it.

If facility can be obtained in the analytical methods as applied to the circle, it will be found that all the latter part of our subject becomes easy. The student is recommended then to dwell on this part of the subject and to master it thoroughly before passing on to the more general conic sections.

76. The equation of the tangent to the circle $x^2+y^2=a^2$ at the point (x_1, y_1) can be found by exactly the same method as that which we have employed in § 74 and the student is advised to go through the work step by step for himself and to obtain the equation of the tangent at (x_1, y_1) in the standard form

$$xx_1+yy_1=a^2.$$

This could of course be got from the more general form we had before by writing $g=0, f=0$ and $c=-a^2$.

77. Condition for tangency.

To find the condition that the line $y=mx+c$ should be a tangent to the circle $x^2+y^2=a^2$.

We treat the two equations as simultaneous so as to find the points common to the line and the circle. On eliminating y we get

$$x^2+(mx+c)^2=a^2,$$

that is $$x^2(1+m^2)+2mcx+c^2-a^2=0.$$

Now if the line be a tangent to the circle, the two values of x furnished by this equation must be equal, otherwise the line would meet the circle in two points not coincident.

The condition then for tangency is

$$m^2c^2 = (c^2 - a^2)(1 + m^2)$$

which reduces to $$c^2 = a^2(1 + m^2).$$

Thus we see that the two lines

$$y = mx \pm a\sqrt{1 + m^2}$$

are both tangents to the circle.

78. It follows from the preceding article that the lines

$$y - k = m(x - h) \pm a\sqrt{1 + m^2}$$

are both of them tangents to the circle whose equation is

$$(x - h)^2 + (y - k)^2 = a^2.$$

For if we eliminate $y - k$ between the equations

$$y - k = m(x - h) + a\sqrt{1 + m^2} \quad \text{..............(1)}$$

and $$(x - h)^2 + (y - k)^2 = a^2 \quad \text{..................(2)}$$

so as to find where the line meets the circle we shall obtain exactly the same quadratic equation in $x - h$ as we should have had in x if we had eliminated y between

$$y = mx + a\sqrt{1 + m^2}$$

and $$x^2 + y^2 = a^2,$$

but this resulting quadratic equation in x we know to have equal roots by the last article.

Therefore the quadratic equation in $x-h$ got by eliminating $y - k$ between (1) and (2) has equal roots. That is we get only one value of $x - h$ and therefore one value of x, and then y is determined uniquely from (1). Thus the line (1) meets the circle (2) in two coincident points. That is to say it is a tangent. Similarly also

$$y - k = m(x - h) - a\sqrt{1 + m^2}$$

is a tangent.

Examples. 1. Prove analytically that the tangent to a circle at any point of it is perpendicular to the radius to that point.

2. Find the equations of the two tangents to the circle $x^2+y^2=a^2$ which make an angle of 60° with the x-axis.

3. Shew that the tangent to the circle $x^2+y^2+2gx+2fy=0$ at the origin is $gx+fy=0$.

4. Shew that the line $x+y=2$ touches the circles $x^2+y^2=2$ and $+y^2+3x+3y-8=0$ at the same point.

5. If $Ax+By+C=0$ be a tangent to the circle $x^2+y^2+2gx+2fy+c=0$, t will also be a tangent to the circle

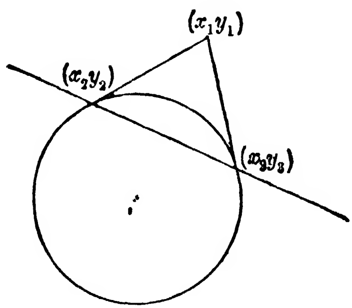
$$x^2+y^2+2gx+2fy+c+\lambda(Ax+By+C)=0.$$

79. Chord of contact.

To find the equation of the chord of contact of tangents from (x_1, y_1) to the circle $x^2+y^2=a^2$.

When we speak of the 'chord of contact' of tangents from (x_1, y_1) we mean the line through the points of contact of the tangents from (x_1, y_1) to the circle.

Let (x_2, y_2), (x_3, y_3) be the points of contact of the tangents from (x_1, y_1) to the circle.



Now the equation of the tangent at (x_2, y_2) is as we have seen

$$xx_2+yy_2=a^2.$$

Therefore, as this passes through (x_1, y_1) we have

$$x_1x_2+y_1y_2=a^2.$$

Similarly $$x_1x_3+y_1y_3=a^2.$$

Therefore (x_2, y_2) and (x_3, y_3) both satisfy the equation

$$xx_1 + yy_1 = a^2,$$

which represents a line.

Therefore this is the equation of the chord of contact.

Cor. We see from the equation that the chord of contact is perpendicular to the line joining the centre to the point (x_1, y_1) for the equation of this joining line is

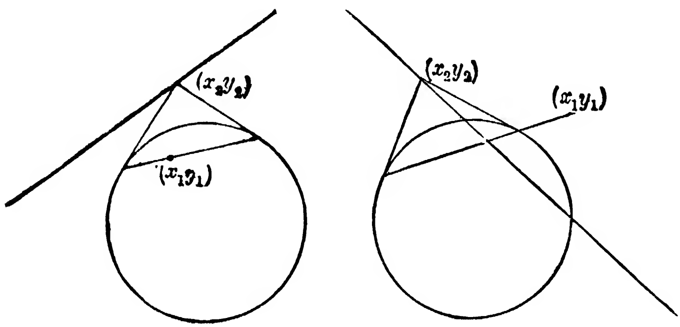
$$\frac{x}{x_1} - \frac{y}{y_1} = 0$$

80. Pole and Polar.

Def. *The polar of a point with respect to a circle* (or conic) *is the locus of the points of intersection of tangents drawn at the extremities of chords through that point.*

To find the equation of the polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$.

Let any chord be drawn through (x_1, y_1) and let the tangents at its extremities meet in (x_2, y_2).



The equation of the chord of contact of tangents from (x_2, y_2) is by § 79

$$xx_2 + yy_2 = a^2.$$

But this goes through (x_1, y_1),

$$\therefore \; x_1x_2 + y_1y_2 = a^2.$$

Therefore the locus of (x_2, y_2) is

$$xx_1 + yy_1 = a^2.$$

We thus see that the polar of (x_1, y_1) as defined above is a line, and it is perpendicular to the line joining the centre to the point (x_1, y_1).

The point (x_1, y_1) is called the *pole* with respect to the line $xx_1 + yy_1 = a^2$.

The terms 'pole' and 'polar' are thus correlative.

The term 'pole' as used in this connection has nothing to do with the same term as used in polar coordinates.

81. If we wish to find the pole of a line whose equation is

$$Ax + By + C = 0,$$

we suppose (x_1, y_1) to be the pole.

Therefore the given line is identical with

$$xx_1 + yy_1 = a^2;$$

$$\therefore \frac{x_1}{A} = \frac{y_1}{B} = \frac{a^2}{-C}.$$

Thus x_1 and y_1 are found.

82. On the definition of Polars. We see from §§ 79, 80 that the polar of a point P with respect to a circle coincides with the chord of contact of tangents from that point, if P be outside the circle. The same is of course true if the point P be inside the circle. For we have then two imaginary tangents from P which will have imaginary points of contact with the circle. These imaginary points of contact lie on the real line $xx_1 + yy_1 = a^2$, as is obvious from the fact that the algebra of § 79 takes no account of whether the points of contact (x_2, y_2) (x_3, y_3) of the tangents are real or imaginary.

We see then that the chord of contact of tangents from a point and the polar of a point are the same line.

Some writers indeed *define* the polar of a point as the chord of contact of tangents real or imaginary from that

point. This is perfectly legitimate. But we prefer the definition we have given in § 80, as then the polar is defined by means of a geometrical property without any reference to imaginary points.

Suppose the polar of a point is defined as the chord of contact, then it is still necessary to shew that *as a result of this definition* the polar of the point is the locus of the points of intersection of tangents drawn at the extremities of chords through the point.

Examples. 1. Shew exactly as in § 79 that the chord of contact of tangents from (x_1, y_1) to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

2. Shew that the polar (as defined in § 80) of the point (x_1, y_1) with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

3. Prove that the points of contact of the tangents from (x_1, y_1) to the circle $x^2+y^2=a^2$ are

$$\left(\frac{a^2x_1+ay_1\sqrt{x_1^2+y_1^2-a^2}}{x_1^2+y_1^2},\ \frac{a^2y_1-ax_1\sqrt{x_1^2+y_1^2-a^2}}{x_1^2+y_1^2}\right)$$

and

$$\left(\frac{a^2x_1-ay_1\sqrt{x_1^2+y_1^2-a^2}}{x_1^2+y_1^2},\ \frac{a^2y_1+ax_1\sqrt{x_1^2+y_1^2-a^2}}{x_1^2+y_1^2}\right).$$

Write down the equation of the line through these points and shew that it reduces to $xx_1+yy_1=a^2$.

4. Prove the following geometrical construction for the polar of a point P with respect to a circle whose centre is C: Join CP and take Q on CP such that $CQ \cdot CP=$ square of radius, then the line through Q perpendicular to CP is the polar.

83. The student will have noticed that the *form* of the equation of the tangent to a circle at (x_1, y_1), when (x_1, y_1) is on the circle, is exactly the same as that of the chord of contact of tangents from (x_1, y_1) and of the polar of (x_1, y_1) when the point is not on the circumference of the circle. This makes these equations easy to remember, for we have only one

form to remember for all three. As will be seen later exactly the same property holds for the conic sections in general.

84. Conjugate points.

PROP. *If the polar of a point A goes through B, then the polar of B goes through A.*

Take the origin at the centre of the circle so that the equation is

$$x^2 + y^2 = a^2.$$

Let (x_1, y_1) be the coordinates of A, and (x_2, y_2) of B. The polar of A is then

$$xx_1 + yy_1 = a^2.$$

If this passes through (x_2, y_2) we have

$$x_2x_1 + y_2y_1 = a^2$$

which is the condition also that (x_1, y_1) should lie on the line

$$xx_2 + yy_2 = a^2.$$

But this is the polar of (x_2, y_2).

Thus the proposition is proved.

Two points such that each lies on the polar of the other are called conjugate points.

Thus the condition that (x_1, y_1) and (x_2, y_2) should be conjugate points for the circle $x^2 + y^2 = a^2$ is $x_1x_2 + y_1y_2 = a^2$. It is quite easy to prove that the condition that (x_1, y_1) and (x_2, y_2) should be conjugate points for the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

Conjugate lines. To find the condition that the pole of the line

$$lx + my + n = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with respect to the circle $x^2 + y^2 = a^2$ should lie on the line

$$l'x + m'y + n' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let (x_1, y_1) be the pole of (1),

$$\therefore\ xx_1 + yy_1 = a^2$$

is identical with (1);

$$\therefore\ \frac{x_1}{l} = \frac{y_1}{m} = \frac{-a^2}{n}.$$

If then (x_1, y_1) lies on (2)

$$-\frac{ll'a^2}{n} - \frac{mm'a^2}{n} + n' = 0,$$

$$\therefore\ ll' + mm' - \frac{nn'}{a^2} = 0.$$

It follows from the symmetrical nature of this relation which is unaltered by the interchange of l, m, n with l', m', n' respectively that if the pole of (1) lies on (2) then the pole of (2) will lie on (1).

Two lines such that each contains the pole of the other are called conjugate lines.

The student need not trouble to remember this condition for conjugate lines as it is much easier to commit to memory the formula in the general case which will be considered in a subsequent chapter.

Equation of chord in terms of its middle point.

85. *If (x_1, y_1) be the middle point of a chord of the circle $x^2 + y^2 = a^2$, the equation of that chord is*

$$xx_1 + yy_1 = x_1^2 + y_1^2.$$

Let the equation of the chord be

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

θ being its inclination to the axis and r the algebraical distance of (x, y) from (x_1, y_1).

Substitute

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta$$

into the equation of the circle.

$$\therefore\ (x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 = a^2,$$

$$\therefore\ r^2 + 2r(x_1\cos\theta + y_1\sin\theta) + x_1^2 + y_1^2 - a^2 = 0.$$

Now since (x_1, y_1) is the middle point of the chord the values of r furnished by this equation must be equal in magnitude and opposite in sign.

$$\therefore\ x_1 \cos\theta + y_1 \sin\theta = 0,$$

or
$$x_1 \cos\theta = -y_1 \sin\theta.$$

Therefore the equation of the chord is

$$\frac{x - x_1}{\cos\theta}(x_1 \cos\theta) = \frac{y - y_1}{\sin\theta}(-y_1 \sin\theta),$$

that is
$$xx_1 + yy_1 = x_1^2 + y_1^2.$$

86. *To find the middle point of the chord of the circle $x^2+y^2=a^2$ the equation of the line of the chord being $x\cos a+y\sin a-p=0$.*

Let (x_1, y_1) be the middle point of the chord,

$\therefore$ the equation of the line of the chord is

$$xx_1+yy_1=x_1^2+y_1^2.$$

This then must be identical with

$$x\cos a+y\sin a=p,$$

$$\therefore\ \frac{x_1}{\cos a}=\frac{y_1}{\sin a}=\frac{x_1^2+y_1^2}{p}=\lambda \quad \text{(say)}$$

$$\therefore\ x_1=\lambda\cos a, \quad y_1=\lambda\sin a,$$

$$\therefore\ \frac{\lambda^2(\cos^2 a+\sin^2 a)}{p}=\lambda, \qquad \therefore\ \lambda=p.$$

Thus the middle point of the chord is $(p\cos a,\ p\sin a)$.

Similarly the middle point of the chord lying along the line $Ax+By=C$ is given by

$$\frac{x_1}{A}=\frac{y_1}{B}=\frac{x_1^2+y_1^2}{C}=\lambda \quad \text{(say)}$$

$$\therefore\ x_1=\lambda A, \quad y_1=\lambda B$$

and
$$\frac{\lambda^2(A^2+B^2)}{C}=\lambda, \qquad \therefore\ \lambda=\frac{C}{A^2+B^2};$$

$$\therefore \text{ the middle point of the chord is } \left(\frac{AC}{A^2+B^2},\ \frac{BC}{A^2+B^2}\right).$$

87. Geometrical properties.

(i) *The line joining the middle point of a chord of a circle to the centre of the circle is perpendicular to the chord.*

For if (x_1, y_1) be the middle point of a chord of the circle $x^2+y^2=a^2$ the equation of the chord is

$$xx_1+yy_1=x_1^2+y_1^2.$$

But this line is clearly at right angles to

$$\frac{x}{x_1} - \frac{y}{y_1} = 0$$

which is the line through the origin, that is the centre of the circle, and the middle point of the chord.

(ii) *The locus of the middle points of a series of parallel chords is a line through the centre of the circle.*

For if (x_1, y_1) be the middle point of one of two series of chords, the equation of that particular chord is

$$xx_1 + yy_1 = x_1^2 + y_1^2.$$

But the chords being all parallel, have the same 'm,'

$$\therefore \ -\frac{x_1}{y_1} = \text{a constant.}$$

That is the locus of (x_1, y_1) is $\frac{y}{x} = \text{a constant}$, which is a line through the centre.

88. The student should for practice prove by the method of § 85 that the equation of the chord of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1,$$

if (x_1, y_1) be the middle point of the chord.

It should be observed that if $T = 0$ be the equation of the tangent at (x_1, y_1) where

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c,$$

then $T = T_1$ is the form of the equation of the chord whose middle point is (x_1, y_1), T_1 being what T becomes when x_1, y_1 are written for x and y. It will be found later that this is a property applicable to conic sections in general.

89. Trigonometrical notation. If θ be the angle which the radius to a point P on a circle (radius a) makes with the axis of x, the centre being the origin, the coordinates of P are given by

$$x = a\cos\theta, \qquad y = a\sin\theta$$

in whatever quadrant P lies.

It is sometimes convenient to represent points on the circle $x^2+y^2=a^2$ in this way instead of by (x_1, y_1) etc.

The equation of the chord joining the two points

$$(a\cos\theta,\ a\sin\theta),\ (a\cos\phi,\ a\sin\phi)$$

is
$$\frac{x-a\cos\theta}{a(\cos\theta-\cos\phi)}=\frac{y-a\sin\theta}{a(\sin\theta-\sin\phi)},$$

that is
$$(x-a\cos\theta)\,2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$$
$$=-(y-a\sin\theta)2\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2},$$

that is, on dividing by $\sin\dfrac{\theta-\phi}{2}$,

$$x\cos\frac{\theta+\phi}{2}+y\sin\frac{\theta+\phi}{2}=a\cos\frac{\theta-\phi}{2}.$$

The equation of the tangent at $(a\cos\theta,\ a\sin\theta)$ is got at once by making the point (ϕ) move up to and coincide with (θ). We thus get as equation of tangent at (θ)

$$x\cos\theta+y\sin\theta=a.$$

This could of course be got immediately from the tangent at (x_1, y_1), viz.

$$xx_1+yy_1=a^2$$

by writing
$$x_1=a\cos\theta,\quad y_1=a\sin\theta.$$

90. We could by analysis prove all the geometrical properties of a circle. It must not, however, be supposed that these properties are in all cases more easily proved by analysis than by pure geometry. Sometimes the methods of analysis are short and simple; but there are cases where they are complicated and inferior to the methods of pure geometry.

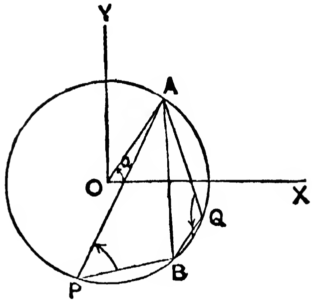
Suppose we were required to prove by analysis that angles in the same segment of a circle are equal. We could proceed as follows.

Let AB be a chord of the circle radius a.

Take the origin O at the centre of the circle, and the line perpendicular to AB as the axis of x.

Let P be a point on the arc of the larger segment cut off by AB, Q a point on the arc of the smaller segment.

Let the points P and Q be expressed $(a\cos\theta,\ a\sin\theta)$, $(a\cos\phi,\ a\sin\phi)$ respectively.



Let the point A be $(a\cos\alpha,\ a\sin\alpha)$ so that B is $\{a\cos(-\alpha),\ a\sin(-\alpha)\}$.

Therefore the equation of AP joining (θ) and (α) is

$$x\cos\frac{\theta+\alpha}{2}+y\sin\frac{\theta+\alpha}{2}=a\cos\frac{\theta-\alpha}{2}.$$

And the equation of BP joining (θ) and $(-\alpha)$ is

$$x\cos\frac{\theta-\alpha}{2}+y\sin\frac{\theta-\alpha}{2}=a\cos\frac{\theta+\alpha}{2}.$$

The 'm' of the line AP is therefore $-\cot\dfrac{\theta+\alpha}{2}$

and the 'm' of the line BP is $-\cot\dfrac{\theta-\alpha}{2}$;

$$\therefore \tan BPA=\frac{-\cot\dfrac{\theta+\alpha}{2}+\cot\dfrac{\theta-\alpha}{2}}{1+\cot\dfrac{\theta+\alpha}{2}\cot\dfrac{\theta-\alpha}{2}}=\frac{\sin\left(\dfrac{\theta+\alpha}{2}-\dfrac{\theta-\alpha}{2}\right)}{\cos\left(\dfrac{\theta+\alpha}{2}-\dfrac{\theta-\alpha}{2}\right)}=\frac{\sin\alpha}{\cos\alpha}=\tan\alpha,$$

$$\therefore\ \angle BPA=\tfrac{1}{2}\angle BOA.$$

Similarly the 'm' of QA is $-\cot\dfrac{\phi+\alpha}{2}$

and the 'm' of QB is $-\cot\dfrac{\phi-\alpha}{2}$;

$$\therefore \tan AQB=\frac{-\cot\dfrac{\phi-\alpha}{2}+\cot\dfrac{\phi+\alpha}{2}}{1+\cot\dfrac{\phi+\alpha}{2}\cot\dfrac{\phi-\alpha}{2}}=\frac{-\sin\alpha}{\cos\alpha}=-\tan\alpha.$$

We have thus proved that the angle $BPA=\frac{1}{2}\angle BOA$ wherever P is on the arc of the larger segment, and the angles AQB, BPA are supplementary independently of the position of Q on the arc of the smaller segment.

All that we have proved could have been proved more easily or quite as easily by pure geometry.

91. Polar equation of circle. It is easily seen that the polar equation of a circle whose radius is a and the polar coordinates of whose centre are (c, α) has for its equation

$$r^2 - 2rc\cos(\theta - \alpha) + c^2 = a^2$$

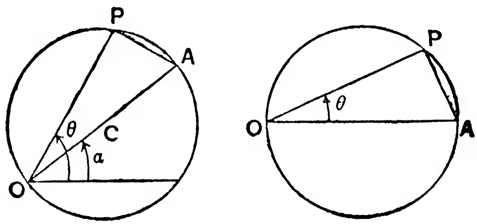
for the left-hand side is the square of the distance of the point (r, θ) from the point (c, α).

When the pole is on the circumference of the circle $c = a$ and the polar equation becomes

$$r = 2a\cos(\theta - \alpha)$$

which when the initial line is a diameter assumes the simpler form

$$r = 2a\cos\theta.$$



These last equations can be got quite easily from figures for if OA be the diameter through the pole O, and P any point on the circle

$$OP = OA\cos AOP.$$

Systems of Circles.

92. Orthogonal circles.

The necessary and sufficient condition that the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0,$$

should cut orthogonally is

$$2gg' + 2ff' = c + c'.$$

[Def. Two circles are said to cut orthogonally when the tangents at their points of intersection are at right angles.]

Let A and B be the centres of the two circles, P a point of intersection.

Then if the tangents at P to the two circles are at right angles, these tangents being at right angles to the radii, APB must be a right angle,

$$AB^2 = AP^2 + BP^2.$$

Now the coordinates of A are $(-g, -f)$ and $AP^2 = g^2 + f^2 - c$. And the coordinates of B are $(-g', -f')$ and $BP^2 = g'^2 + f'^2 - c'$.

$$\therefore\ (g-g')^2 + (f-f')^2 = g^2 + f^2 - c + g'^2 + f'^2 - c',$$

$$\therefore\ 2gg' + 2ff' = c + c'.$$

This condition then is necessary, and it can be seen to be sufficient by working the algebra backwards.

93. Radical axis.

The locus of points from which tangents to two given circles are equal is a straight line perpendicular to the line joining their centres and passing through the points of intersection of the circles.

Let the two circles have for their equations

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \ldots\ldots\ldots\ldots\ldots(2).$$

Let $P\ (x, y)$ be a point from which the tangents to (1) and (2) are equal.

Now square of tangent from P to (1) is $x^2 + y^2 + 2gx + 2fy + c$. And square of tangent from P to (2) is $x^2 + y^2 + 2g'x + 2f'y + c'$.

$$\therefore\ x^2 + y^2 + 2gx + 2fy + c = x^2 + y^2 + 2g'x + 2f'y + c',$$

$$\therefore\ 2(g-g')x + 2(f-f')y + c - c' = 0 \ldots\ldots\ldots(3).$$

Therefore the locus of (x, y) is a straight line; and points which satisfy (1) and (2) simultaneously also satisfy (3); therefore (3) goes through the points of intersection of (1) and (2).

Moreover the coordinates of the centres of (1) and (2) are $(-g, -f)$, $(-g', -f')$ respectively.

Therefore the 'm' of the line joining the centres is $\dfrac{f-f'}{g-g'}$.

But the 'm' of (3) is $-\dfrac{g-g'}{f-f'}$, and the product of these is -1.

Therefore (3) is perpendicular to the line joining the centres the circles.

The line (3) is called the *radical axis* of the two circles (1) and (2).

COR. 1. *The three radical axes of three circles taken in pairs meet in a point* (called the *radical centre* of the three circles).

For if
$$x^2+y^2+2gx+2fy+c=0 \quad \text{..............(1)},$$
$$x^2+y^2+2g'x+2f'y+c'=0 \quad \text{..............(2)},$$
$$x^2+y^2+2g''x+2f''y+c''=0 \quad \text{..............(3)},$$
be the equations of the circles.

The radical axis of (2) and (3) is
$$2(g'-g'')x+2(f'-f'')y+c'-c''=0 \text{.........(4)}.$$
The radical axis of (3) and (1) is
$$2(g''-g)x+2(f''-f)y+c''-c=0 \text{.........(5)},$$
and of (1) and (2) is
$$2(g-g')x+2(f-f')y+c-c'=0 \text{.........(6)}.$$
Now (4) + (5) + (6) is an identity.

Therefore the three equations (4), (5) and (6) hold simultaneously, that is, the radical axes meet in a point.

COR. 2. *If two circles touch one another their radical axis is their common tangent at the point of contact.*

94. Equation of two circles.

By choosing as the axis of x the line joining the centres of two circles, and as the axis of y their radical axis, the equations of the circles are of the simple form
$$x^2+y^2+2gx+c=0,$$
$$x^2+y^2+2g'x+c=0.$$

For since the y coordinate of each centre is zero the circles have their equations of the form

$$x^2 + y^2 + 2gx + c = 0,$$
$$x^2 + y^2 + 2g'x + c' = 0.$$

The radical axis of these is

$$2(g - g')x + c - c' = 0.$$

But the radical axis is $\quad x = 0.$

$$\therefore \ c' = c.$$

Thus the proposition is proved.

95. PROP. *The difference of the square of the tangents to two circles from any point in their plane varies as the distance of the point from their radical axis.*

(Let x_1, y_1) be any point in the plane of the two circles

$$x^2 + y^2 + 2gx + c = 0,$$
$$x^2 + y^2 + 2g'x + c = 0.$$

The difference of the squares of the tangents to these from (x_1, y_1) is

$$(x_1^2 + y_1^2 + 2gx_1 + c) - (x_1^2 + y_1^2 + 2g'x + c)$$
$$= 2x_1(g - g'),$$

which varies as x_1, which is the distance of the point from the axis of y, that is the radical axis.

96. **Coaxial circles.** A system of circles such that the radical axis for *any* pair of them is the same is called a *coaxial system.*

Clearly such circles will all have their centres along the same line. Taking this line of centres as the axis of x and the common radical axis for the axis of y the equations of the circles will be of the form

$$x^2 + y^2 + 2gx + c = 0,$$
$$x^2 + y^2 + 2g'x + c = 0,$$
$$x^2 + y^2 + 2g''x + c = 0.$$
$$\text{etc.} \qquad \text{etc.}$$

Belonging to this system will be the two circles

$$x^2 + y^2 + 2\sqrt{c}x + c = 0 \text{ ....(1)}$$

and
$$x^2 + y^2 - 2\sqrt{c}x + c = 0 \text{ ....(2),}$$

that is
$$(x + \sqrt{c})^2 + y^2 = 0$$

and
$$(x - \sqrt{c})^2 + y^2 = 0,$$

that is two circles of zero radius with their centres at

$$(-\sqrt{c}, 0)\ (+\sqrt{c}, 0).$$

The centres of these two circles of zero radius, point circles as they are sometimes called, are known as the *limiting points* of the system of coaxial circles.

The limiting points, lying on the line of centres, are real if c be positive, otherwise they are imaginary.

97. Limiting points.

PROP. *If L and L' be the limiting points of a system of coaxial circles the polar of either of these points with respect to any circle of the system passes through the other.*

For taking the line of centres as axis of x, and the radical axis as that of y, the equation of any circle of the system is of the form

$$x^2 + y^2 + 2gx + c = 0.$$

The coordinates of L are $(\sqrt{c}, 0)$ and of L' $(-\sqrt{c}, 0)$. (§ 96)

The polar of L $(\sqrt{c}, 0)$ with respect to the circle is

$$x\sqrt{c} + y\,(0) + g\,(x + \sqrt{c}) + c = 0,$$

that is
$$(\sqrt{c} + g)(x + \sqrt{c}) = 0.$$

But
$$g \neq -\sqrt{c},$$

$$\therefore\ x + \sqrt{c} = 0,$$

which is a line through L' parallel to the radical axis.

98. PROP. *All the circles of a coaxial system are cut orthogonally by every circle passing through the limiting points.*

Take the axes as before.

The equation of a circle of the system is of the form

$$x^2 + y^2 + 2gx + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and the equation of a circle passing through $(\sqrt{c}, 0)$, $(-\sqrt{c}, 0)$ is of the form

$$x^2 + y^2 + 2fy - c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

since the x coordinate of its centre is zero and when

$$y = 0, \quad x = \pm\sqrt{c}.$$

The condition that (1) and (2) should cut orthogonally is

$$2g(0) + 2(0)f = c - c \quad (\S\ 92)$$

which is satisfied.

99. *The general equation of circles coaxial with two given circles*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

is $S + kS' = 0$, where k is any constant.

For clearly $S + kS' = 0$ for any constant value of k is a circle, since the coefficient of x^2 = coefficient of y^2, and the coefficient of xy is zero.

Moreover the locus $S + kS' = 0$ passes through the points common to the two given circles.

Thus for different values of k it represents circles coaxial with the given circles.

100. *If* $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots\ldots(1)$

be a circle and $\quad L \equiv Ax + By + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$

a line, the equation

$$S - kL = 0$$

for any constant value of k will be a circle cutting the given circle in the points where it is cut by the line.

For it is clear that

$$S - kL \equiv x^2 + y^2 + 2gx + 2fy + c - k(Ax + By + C) = 0$$

represents a circle and it is satisfied by points which satisfy (1) and (2) simultaneously.

COR. If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$
be some circle, then the equation of *any* other circle can be expressed in the form

$$S + (lx + my + n) = 0.$$

EXAMPLES.

1. The condition that the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

should touch is

$$(2gg' + 2ff' - c - c')^2 = 4(g^2 + f^2 - c)(g'^2 + f'^2 - c').$$

2. The locus of points, the tangents from which to two given circles are in a given ratio is a circle coaxial with the given circles.

[Use § 94.]

3. The locus of points such that the difference of the squares of the tangents from them to two given circles is constant is a line parallel to their radical axis.

4. The equation of the circle orthogonal to the three circles

$$x^2 + y^2 + 2d_1x + 2e_1y + f_1 = 0$$
$$x^2 + y^2 + 2d_2x + 2e_2y + f_2 = 0$$
$$x^2 + y^2 + 2d_3x + 2e_3y + f_3 = 0$$

is

$$\begin{vmatrix} x^2 + y^2, & x, & y, & 1 \\ -f_1, & d_1, & e_1, & -1 \\ -f_2, & d_2, & e_2, & -1 \\ -f_3, & d_3, & e_3, & -1 \end{vmatrix} = 0.$$

5. The equation of the circle circumscribing the triangle formed by the lines

$$Ax + By + C = 0, \quad A'x + B'y + C' = 0, \quad A''x + B''y + C'' = 0$$

is

$$\begin{vmatrix} \dfrac{A^2 + B^2}{Ax + By + C}, & A, & B \\ \dfrac{A'^2 + B'^2}{A'x + B'y + C'}, & A', & B' \\ \dfrac{A''^2 + B''^2}{A''x + B''y + C''}, & A'', & B'' \end{vmatrix} = 0.$$

[Clearly

$$\lambda(A'x+B'y+C')(A''x+B''y+C'')+\mu(A''x+B''y+C'')(Ax+By+C)$$
$$+\nu(Ax+By+C)(A'x+B'y+C')=0,$$

will pass through the vertices of the triangle formed by the given lines, for this equation is satisfied by any two of the given equations taken simultaneously. Choose λ, μ, ν so that this is a circle.]

6. The condition that the circle circumscribing the triangle formed by the three lines

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0$$
$$a_3x+b_3y+c_3=0$$

should have its centre on the x-axis is

$$\begin{vmatrix} a_2a_3-b_2b_3, & a_3a_1-b_3b_1, & a_1a_2-b_1b_2 \\ a_3b_2+a_2b_3, & a_1b_3+a_3b_1, & a_2b_1+a_1b_2 \\ b_3c_2+b_2c_3, & b_1c_3+b_3c_1, & b_2c_1+b_1c_2 \end{vmatrix}=0.$$

7. The straight line $x\cos\alpha+y\sin\alpha=p$ being denoted by (αp), find the equation of the circle circumscribing the triangle formed by the lines (αp), (βq), (γr) and shew that if it passes through the origin then

$$qr\sin(\beta-\gamma)+rp\sin(\gamma-\alpha)+pq\sin(\alpha-\beta)=0.$$

8. Shew that the circle on the chord $x\cos\alpha+y\sin\alpha-p=0$ of the circle $x^2+y^2-a^2=0$ as diameter is

$$x^2+y^2-a^2-2p(x\cos\alpha+y\sin\alpha-p)=0.$$

[Use § 100.]

9. If two circles cut a third circle orthogonally, the radical axis of the two circles passes through the centre of the third circle.

10. Shew that if $S_1=0$, $S_2=0$, $S_3=0$ be the equations of three circles of which each two cut orthogonally, the equation

$$l_1S_1+l_2S_2+l_3S_3=0$$

represents a real circle except in certain cases where it represents a straight line.

11. Shew that the condition that the two circles

$$a(x^2+y^2)+gx+fy+c=0$$

and
$$a'(x^2+y^2)+g'x+f'y+c'=0$$

may touch each other is

$$(af'-a'f)(cf'-c'f)+(ag'-a'g)(cg'-c'g)+(ac'-a'c)^2=\tfrac{1}{4}(fg'-f'g)^2.$$

12. Shew that if two points are conjugate with respect to a circle the square of the distance between them is equal to the sum of the squares of the tangents from them to the circle.

13. If two points P and Q are conjugate with respect to a circle S the circle on PQ as diameter cuts S at right angles.

14. Shew that the general equation of all circles cutting at right angles the circles represented by

$$x^2+y^2-2a_1x-2b_1y+c_1=0, \quad x^2+y^2-2a_2x-2b_2y+c_2=0$$

is
$$\begin{vmatrix} x^2+y^2, & x, & y \\ c_1, & a_1, & b_1 \\ c_2, & a_2, & b_2 \end{vmatrix} + k \begin{vmatrix} x, & y, & 1 \\ a_1, & b_1, & 1 \\ a_2, & b_2, & 1 \end{vmatrix} = 0.$$

15. Find the equation of the circle whose diameter is the common chord of the circles

$$x^2+y^2+2x+3y+1=0 \text{ and } x^2+y^2+4x+3y+2=0.$$

16. Find the coordinates of the limiting points of the circles

$$x^2+y^2+2x+4y+7=0, \quad x^2+y^2+4x+2y+5=0.$$

17. Find the equation of the circle to which the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is self-conjugate.

[A triangle is self-conjugate for a circle when each pair of its vertices are conjugate points.]

18. Prove that the circumcircle of the triangle formed by the lines

$$x\cos\alpha+y\sin\alpha=a\sec\alpha+b\sin\alpha,$$
$$x\cos\beta+y\sin\beta=a\sec\beta+b\sin\beta,$$
$$x\cos\gamma+y\sin\gamma=a\sec\gamma+b\sin\gamma,$$

passes through the points $(0, b)$.

19. The polars of a point P with respect to two given circles meet in Q; shew that the radical axis of the circles bisects PQ.

[Use § 94.]

20. Circles are drawn through the point $(c, 0)$ touching the circle $x^2 + y^2 = a^2$. Shew that the locus of the pole of the axis of x with respect to these circles is the curve

$$4a^2(x-c)^4 = (a^2 - c^2)\{a^2 - (c - 2x)^2\}y^2.$$

21. The straight line $lx + my - 1 = 0$ meets the lines

$$ax^2 + 2hxy + by^2 = 0$$

in the points P and Q; shew that the equation of the circle described on PQ as diameter is

$$(x^2 + y^2)(am^2 - 2hlm + bl^2) - 2x(bl - hm) - 2y(am - hl) + a + b = 0.$$

22. Prove that if $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$ be four circles of which each pair is orthogonal, their equations being in the form in which S denotes the square of the tangent from any point, then the condition that

$$\lambda S_1 + \mu S_2 + \nu S_3 + \rho S_4 = 0 \quad \text{and} \quad \lambda' S_1 + \mu' S_2 + \nu' S_3 + \rho' S_4 = 0$$

should be orthogonal is

$$\lambda\lambda' r_1^2 + \mu\mu' r_2^2 + \nu\nu' r_3^2 + \rho\rho' r_4^2 = 0$$

where r_1, r_2, r_3, r_4 are the radii of the circles.

23. The length of the common chord of the two circles

$$x^2 + y^2 + 2\lambda x + c = 0,$$
$$x^2 + y^2 + 2\mu y - c = 0$$

is
$$2\sqrt{(\lambda^2 - c)(\mu^2 + c)/(\lambda^2 + \mu^2)}.$$

24. If the origin be at one of the limiting points of a system of coaxial circles of which

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is a member, the equation of the system of circles cutting them all orthogonally is

$$(x^2 + y^2)(g + \mu f) + c(x + \mu y) = 0.$$

25. Two circles of radii R_1, R_2 with their centres at a distance $d > (R_1 + R_2)$ may, by properly choosing the axes of coordinates in one of two different ways, be represented by the equations

$$x^2 + y^2 - 2ax + c^2 = 0, \qquad x^2 + y^2 - 2by + c^2 = 0,$$

where
$$a^2 = \tfrac{1}{2}(R_1^2 - R_2^2 + d^2), \quad b^2 = \tfrac{1}{2}(R_2^2 - R_1^2 + d^2)$$

and
$$c^2 = \tfrac{1}{2}(d^2 - R_1^2 - R_2^2).$$

26. From a fixed point (ξ, η) perpendiculars are drawn to the straight lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Shew that the equation of the circle circumscribing the quadrilateral so formed is

$$(ab - h^2)\{x(x-\xi) + y(y-\eta)\} - (hf - bg)(x - \xi) - (gh - af)(y - \eta) = 0.$$

27. Circles are drawn with their centres on the axis of x and touching the straight line $y = x \tan \alpha$. Shew that the points of contact of tangents from a fixed point (h, k) will lie on the curve given by

$$(x-h)^2(y^2 - x^2 \sin^2 \alpha) - 2xy(x-h)(y-k)\sin^2\alpha + y^2\cos^2\alpha(y-k)^2 = 0.$$

28. If four points P, Q, R, S be taken, and the square of the tangent from P to the circle on QR as diameter be denoted by (P, QR), then

$$(P, RS) - (P, QS) - (Q, RS) + (Q, PR) = 0.$$

29. Shew that with respect to the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$, $y = k$, the equation of the pedal line of the point (other than the origin) where the circumcircle cuts the axis of x is

$$a(a-b)x + 2ahy = 2bhk.$$

30. The centres C_1, C_2, C_3, C_4 of four circles form a parallelogram, C_1 and C_3 being opposite vertices. Prove that the locus of a point such that the lengths t_1, t_2, t_3, t_4 of the tangents drawn from it to these four circles obey the relation $t_1t_3 = t_2t_4$ is a curve of the second degree.

31. Shew that the general equation of a circle which touches the two circles

$$x^2 + y^2 + c^2 + 2ax = 0, \qquad x^2 + y^2 + c^2 + 2bx = 0,$$

may be written in the form

$$\{(c^2+\mu^2)(c^2+ab)\}^{\frac{1}{2}}(x^2+y^2+c^2+2\lambda x) + c(ab-\lambda^2)^{\frac{1}{2}}(x^2+y^2-c^2+2\mu y) = 0,$$

where μ has any value, and λ is either root of the quadratic equation

$$(a+b)(c^2+\lambda^2) = 2\lambda(c^2+ab).$$

CHAPTER VI.

CHANGE OF AXES.

101. Before we pass on to the analytical investigation of the conic sections we shall obtain, for future use, the formulae necessary to express the coordinates of a point in a plane referred to two axes in terms of the coordinates of the same point referred to two other axes.

We have already made use of the fact that if (x, y) be the coordinates of a point P, referred to two axes, (x_1, y_1) the coordinates of a point A referred to the same axes, then the coordinates of P referred to axes through A parallel to the original axes are $(x - x_1, y - y_1)$. Denoting these by (X, Y) we have

$$x = X + x_1, \qquad y = Y + y_1.$$

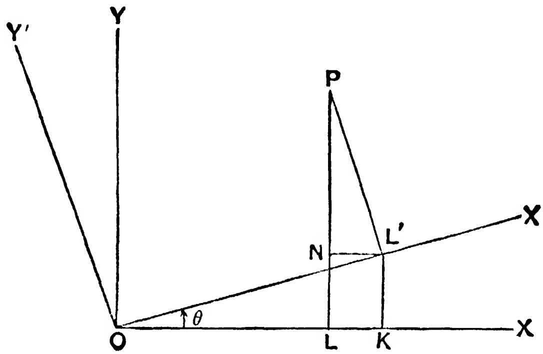
We see then how to obtain the equation of a curve referred to two new axes when its equation is already known referred to two axes to which they are respectively parallel. We have only to write $X + x_1$, $Y + y_1$ for x and y in the given equation, and the new equation in X, Y will be obtained.

This is changing the origin without changing the directions of the axes.

We now pass to the consideration of the problem of a change in the directions of the axes without a change of the origin.

102. Transition from one set of rectangular axes to another with the same origin.

Let (x, y) be the coordinates of a point P referred to rectangular axes OX, OY. Let (x', y') be the coordinates of the same point P referred to rectangular axes OX', OY' with the same origin.



Let $\angle XOX'$ measured in the usual positive direction $= \theta$.

Draw PL, PL' perpendicular to OX and OX' respectively. Draw $L'K$ perpendicular to OX and $L'N$ perpendicular to PL.

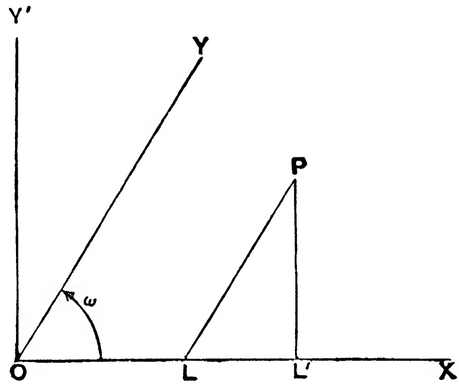
Then
$$x = OL = OK - NL'$$
$$= x' \cos\theta - y' \sin\theta \quad\text{............(1)},$$
$$y = LP = KL' + NP$$
$$= x' \sin\theta + y' \cos\theta \quad\text{............(2)}.$$

From these, or independently, we get
$$x' = x\cos\theta + y\sin\theta,$$
$$y' = -x\sin\theta + y\cos\theta.$$

These latter are not wanted so frequently as the former. They can be always obtained from (1) and (2) by interchanging x and x', y and y' and writing $-\theta$ for θ.

103. Transition from oblique axes to rectangular axes with the same axis of x.

Let (x, y) be the coordinates of P referred to oblique axes OX, OY containing an angle ω; (x', y') the coordinates of the same point referred to rectangular axes OX, OY'.



Draw PL, PL' parallel to OY, OY' to meet the x-axis in L and L'.

Then
$$\left.\begin{aligned} x &= OL = OL' - LL' = x' - y' \cot \omega \\ y &= LP = y' \operatorname{cosec} \omega \end{aligned}\right\}$$

and
$$\left.\begin{aligned} x' &= OL' = OL + LL' = x + y \cos \omega \\ y' &= L'P = y \sin \omega \end{aligned}\right\}.$$

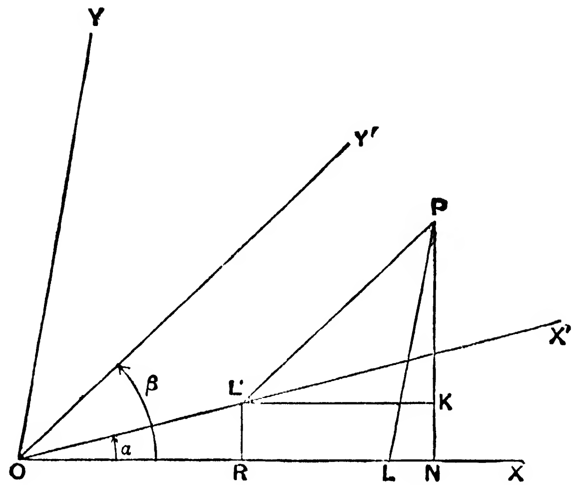
104. Transition from one set of oblique axes to another with the same origin.

Let (x, y) be the coordinates of a point P referred to axes OX, OY containing an angle ω; (x', y') the coordinates of the same point referred to axes OX', OY' containing an angle ω'. Let α and β be the angles which OX' and OY' respectively make with OX.

Draw PL parallel to OY to meet OX in L, so that $OL = x$, $LP = y$.

Draw PL' parallel to OY' to meet OX' in L', so that $OL' = x'$, $L'P = y'$.

Now draw PN perpendicular to OX; $L'R$ and $L'K$ perpendicular to OX and PN respectively.



Then $y \sin \omega = NP = RL' + KP = x' \sin \alpha + y' \sin \beta$.

Similarly, since OX' and OY' make angles $\omega - \alpha$ and $\omega - \beta$ with OY,

$$x \sin \omega = x' \sin (\omega - \alpha) + y' \sin (\omega - \beta).$$

Thus
$$x = x' \frac{\sin (\omega - \alpha)}{\sin \omega} + y' \frac{\sin (\omega - \beta)}{\sin \omega},$$

$$y = x' \frac{\sin \alpha}{\sin \omega} + y' \frac{\sin \beta}{\sin \omega},$$

where $\beta - \alpha = \omega'$.

105. The formulae obtained in the last article are not easily remembered, nor will there be much need to remember them. But it is important to observe that

$$x = kx' + ly',$$
$$y = k'x' + l'y',$$

where k, l, k', l' are constants depending on the angles ω, ω' and α.

The actual formulae of §§ 102, 103 will be wanted from time to time. But they are very easily obtained at any time directly from a figure, so that they need be no burden on the memory.

106. Invariants.

PROP. *If* (x, y) *be the coordinates of a point referred to axes* OX, OY *containing an angle* ω, *and* (x', y') *be the coordinates of the same point referred to axes* OX', OY' *containing an angle* ω', *and if* $ax^2 + 2hxy + by^2$ *in which* a, b, h *are independent of* x *and* y *become* $a'x'^2 + 2h'x'y' + b'y'^2$, *then*

$$\frac{a' + b' - 2h' \cos\omega'}{\sin^2\omega} = \frac{a + b - 2h\cos\omega}{\sin^2\omega}$$

and
$$\frac{a'b' - h'^2}{\sin^2\omega'} = \frac{ab - h^2}{\sin^2\omega}.$$

We have $\quad ax^2 + 2hxy + by^2 = a'x'^2 + 2h'x'y' + b'y'^2$

and $\quad x^2 + 2xy\cos\omega + y^2 = x'^2 + 2x'y'\cos\omega' + y'^2,$

since each of these is the square of the distance of the same point from the origin.

Therefore

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + 2xy\cos\omega + y^2)$$
$$= a'x'^2 + 2h'x'y' + b'y^2 + \lambda(x'^2 + 2x'y'\cos\omega' + y'^2) \ldots (1),$$

for all values of λ.

Now if λ be so chosen that the left-hand side is a perfect square in x and y, viz. $(px + qy)^2$, then since

$$x = kx' + ly',$$
$$y = k'x' + l'y' \qquad (\S\ 105),$$

$$\therefore\ (px + qy)^2 = \{p(kx' + ly') + q(k'x' + l'y')\}^2$$
$$= (p'x' + q'y')^2, \text{ (say)}.$$

Thus whatever value of λ makes the left-hand side of (1) a perfect square in x and y makes the right-hand side a perfect square in x' and y'.

The left-hand side will be a perfect square if

$$(a + \lambda)(b + \lambda) = (h + \lambda \cos \omega)^2,$$

that is if

$$\lambda^2 \sin^2 \omega + \lambda (a + b - 2h \cos \omega) + ab - h^2 = 0.$$

Similarly the right-hand side will be a perfect square if

$$\lambda^2 \sin^2 \omega' + \lambda (a' + b' - 2h' \cos \omega') + a'b' - h'^2 = 0.$$

These two quadratic equations in λ must have exactly the same roots,

$$\therefore \quad \frac{a + b - 2h \cos \omega}{\sin^2 \omega} = \frac{a' + b' - 2h' \cos \omega'}{\sin^2 \omega'}$$

and

$$\frac{ab - h^2}{\sin^2 \omega} = \frac{a'b' - h'^2}{\sin^2 \omega'}.$$

On account of this property $\dfrac{a + b - 2h \cos \omega}{\sin^2 \omega}$ and $\dfrac{ab - h^2}{\sin^2 \omega}$ are called *invariants.*

In the special case where we transform from one set of rectangular axes to another,

$$a + b = a' + b',$$

$$ab - h^2 = a'b' - h'^2.$$

These invariants are of importance in the development of the theory of the general equation of the second degree and the student is recommended to master the proof given of them. It would be well for him to work out the special case of rectangular axes by the same method we have employed in the general case when the axes may be oblique.

107. Removal of the xy term.

PROP. *If (x, y) be the coordinates of a point referred to rectangular axes, it is always possible to transform to rectangular axes with the same origin so that $ax^2 + 2hxy + by^2$ becomes $a'x'^2 + b'y'^2$ in which the term in $x'y'$ is wanting, (x', y') being the same point referred to the new axes.*

For if the axes be turned through an angle θ we have by § 102

$$x = x'\cos\theta - y'\sin\theta,$$
$$y = x'\sin\theta + y'\cos\theta,$$

$$\therefore\ ax^2 + 2hxy + by^2 = a(x'\cos\theta - y'\sin\theta)^2$$
$$+ 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)^2$$
$$= (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\,x'^2$$
$$-2\{(a-b)\sin\theta\cos\theta - h(\cos^2\theta - \sin^2\theta)\}\,x'y'$$
$$+ (a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)\,y'^2.$$

The term in $x'y'$ will disappear if

$$\tfrac{1}{2}(a-b)\sin 2\theta - h\cos 2\theta = 0,$$

that is if
$$\tan 2\theta = \frac{2h}{a-b},$$

and this equation can always be satisfied by a real value of θ.

We have
$$2\theta = n\pi + \tan^{-1}\frac{2h}{a-b},$$

$$\therefore\ \theta = \tfrac{1}{2}\tan^{-1}\frac{2h}{a-b} \quad\text{..................(1)},$$

or
$$\theta = \frac{\pi}{2} + \tfrac{1}{2}\tan^{-1}\frac{2h}{a-b} \quad\text{..................(2)},$$

or
$$\theta = \pi + \tfrac{1}{2}\tan^{-1}\frac{2h}{a-b} \quad\text{..................(3)},$$

or
$$\theta = \frac{3\pi}{2} + \tfrac{1}{2}\tan^{-1}\frac{2h}{a-b} \quad\text{..................(4)}.$$

These all really give the same new axes of coordinates, what is the positive direction of the x-axis in one case being the positive or negative direction of the x- or y-axis in another.

Special case where $ab - h^2 = 0$. In the special case where $ab - h^2 = 0$ the removal of the xy term in

$$ax^2 + 2hxy + by^2$$

by turning the axes through an angle θ given by $\tan\theta = \dfrac{2h}{a-b}$ will make either $a' = 0$ or $b' = 0$.

For $$ax^2 + 2hxy + by^2 \text{ becomes } a'x'^2 + b'y'^2.$$

Now $ab = h^2$ is the condition that the pair of lines through the origin
$$ax^2 + 2hxy + by^2 = 0$$
should be coincident.

Therefore $$a'x'^2 + b'y'^2 = 0$$
represents a pair of coincident straight lines, which is impossible unless $a' = 0$ or $b' = 0$.

The same can also be seen by means of the invariants. For
$$a'b' = ab - h^2 = 0,$$
$$\therefore\ a' = 0 \text{ or } b' = 0.$$

Oblique axes.

108. *To find the condition that the two lines whose equations are*
$$Ax + By + C = 0$$
and $$A'x + B'y + C' = 0$$
should be at right angles, the axes being inclined at an angle ω.

Transform to rectangular axes keeping the origin and the axis of x unchanged.

As in § 103, we have
$$x = x' - y' \cot \omega, \quad y = y' \operatorname{cosec} \omega.$$

Thus the equations of the two lines referred to the new rectangular axes are
$$A(x' - y' \cot \omega) + By' \operatorname{cosec} \omega + C = 0$$
and $$A'(x' - y' \cot \omega) + B'y' \operatorname{cosec} \omega + C' = 0,$$
that is $$A \sin \omega . x' + (B - A \cos \omega) y' + C' \sin \omega = 0$$
and $$A' \sin \omega . x' + (B' - A' \cos \omega) y' + C' \sin \omega = 0.$$

The condition that these should be perpendicular is
$$AA' \sin^2 \omega + (B - A \cos \omega)(B' - A' \cos \omega) = 0$$
that is $$AA' + BB' - (AB' + A'B) \cos \omega = 0.$$

This condition the student has probably already obtained for himself in another way. (See Ex. 2 of Chapter III, p. 43.)

109. *To find the length of the perpendicular from the point $P(x_1, y_1)$ on the line whose equation is*

$$Ax + By + C = 0$$

when the axes are inclined at an angle ω.

Transform to rectangular axes with the same origin and the same axis of x as before.

Use dashed letters for the new coordinates so that

$$\left.\begin{aligned} x &= x' - y' \cot\omega \\ y &= y' \operatorname{cosec}\omega \end{aligned}\right\}, \qquad \left.\begin{aligned} x_1 &= x_1' - y_1' \cot\omega \\ y_1 &= y_1' \operatorname{cosec}\omega \end{aligned}\right\}.$$

The equation of the line thus becomes

$$A(x' - y' \cot\omega) + By' \operatorname{cosec}\omega + C = 0.$$

The perpendicular from (x_1, y_1) on this is (§ 34)

$$\frac{A(x_1' - y_1' \cot\omega) + By_1' \operatorname{cosec}\omega + C}{\sqrt{A^2 + (B \operatorname{cosec}\omega - A\cot\omega)^2}},$$

and this $$= \frac{(Ax_1 + By_1 + C)\sin\omega}{\sqrt{A^2 \sin^2\omega + (B - A\cos\omega)^2}}$$

$$= \frac{(Ax_1 + By_1 + C)\sin\omega}{\sqrt{A^2 - 2AB\cos\omega + B^2}}.$$

110. *To find the angle between the pair of lines*

$$ax^2 + 2hxy + by^2 = 0,$$

the axes being inclined at an angle ω.

We will here make use of invariants.

Transform to new and *rectangular* axes with the same origin and let the equation of the lines become

$$a'x'^2 + 2h'x'y' + b'y'^2 = 0.$$

Now if ϕ be the angle between the lines we know from § 62 that

$$\tan\phi = \frac{2\sqrt{h'^2 - a'b'}}{a' + b'}.$$

But by § 106

$$\frac{h^2 - ab}{\sin^2 \omega} = \frac{h'^2 - a'b'}{\sin^2 \frac{\pi}{2}} = h'^2 - a'b',$$

and
$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} = \frac{a' + b' - 2h'\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}} = a' + b',$$

$$\therefore \tan\phi = \frac{2\sin\omega\sqrt{h^2 - ab}}{a + b - 2h\cos\omega}.$$

And the lines will be at right angles if

$$a + b - 2h\cos\omega = 0.$$

EXAMPLES.

1. If (x, y) and (x', y') be the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux + vy$ where u and v are independent of x and y becomes $u'x' + v'y'$, then

$$u^2 + v^2 = u'^2 + v'^2.$$

2. If (x, y) and (x', y') be the coordinates of the same point referred to two sets of axes with the same origin and $ux + vy$ be transformed to $u'x' + v'y'$, then

$$\frac{u^2 - 2uv\cos\omega + v^2}{\sin^2\omega} = \frac{u'^2 - 2u'v'\cos\omega' + v'^2}{\sin^2\omega'},$$

where ω and ω' are the angles between the axes in the two axes.

3. If (x, y) and (x', y') have the same meaning as in Ex. 2 and if

$$x = kx' + ly',$$
$$y = k'x' + l'y',$$

then
$$(kl' - k'l)\sin\omega = \sin\omega'.$$

4. Shew that the equation $x\cos\alpha + y\sin\alpha = p$, when the axes are turned through an angle α, becomes $x = p$. Interpret this fact.

5. The equation of the bisectors of the angles between the lines

$$ax^2 + 2hxy + by^2 = 0$$

is

$$\begin{vmatrix} ax + hy, & hx + by \\ x + y\cos\omega, & y + x\cos\omega \end{vmatrix} = 0,$$

the axes being inclined at an angle ω.

6. OAB is a fixed triangle having AOB a right angle, OA, OB along the axes of coordinates, $AB = 4c$ and $OAB = \alpha$. A circle is drawn circumscribing the triangle, and from any point on this circle perpendiculars are drawn on the sides; shew that the feet of these perpendiculars lie on the line

$$x\cos\phi + y\sin\phi = 4c\sin\phi\cos\phi\sin(\alpha + \phi).$$

Transfer the origin of coordinates to the point $a\cos\alpha$, $a\sin\alpha$; turn the axes of coordinates through an angle $\frac{\pi}{6} - \frac{\alpha}{3}$ and shew that the equation of the line is now

$$x\cos\psi + y\sin\psi = c\sin 3\psi.$$

7. Prove that the transformation of rectangular axes which converts $\frac{X^2}{p} + \frac{Y^2}{q}$ into $ax^2 + 2hxy + by^2$ will convert $\frac{X^2}{p-\lambda} + \frac{Y^2}{q-\lambda}$ into

$$\frac{ax^2 + 2hxy + by^2 - \lambda(ab - h^2)(x^2 + y^2)}{1 - (a + b)\lambda + (ab - h^2)\lambda^2}.$$

CHAPTER VII.

THE CONIC SECTIONS—GENERAL AND STANDARD EQUATIONS.

111. The conic sections are the curves of projection of a circle by means of a vertex V, not in the plane of the circle, on to another plane. In other words they are the sections of a cone having a circular base. It is not necessary that the cone should be a right circular one, that is that the line joining the vertex to the centre of the circular base should be perpendicular to the base.

The sections of any cone having a circular base by planes parallel to the base will be themselves also circles, and the section of the cone by a plane passing through the vertex will be two straight lines.

If the cone be cut by a plane not through the vertex and not parallel to the base the curve of section will be an ellipse, parabola or hyperbola according to the position of the cutting plane. All these curves alike share what it is convenient to call 'the focus and directrix property.' That is, each of these plane curves is the locus of a point such that its distance from a fixed point (the *focus*) in the plane is e times its distance from a fixed straight line called the *directrix*, e being a constant known as the *eccentricity*. For a parabola $e=1$, for an ellipse $e<1$, and for a hyperbola $e>1$. (*Course of Pure Geometry*, Chap. IX.)

112. Proposition. *Every plane section of a cone having a circular base when referred to Cartesian axes of coordinates in the plane is represented by an equation of the second degree.*

We have already seen that this is so if the section be two straight lines or a circle, for both these are represented by an equation of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If now the section be an ellipse, or parabola, or hyperbola, let e be the eccentricity, (x_1, y_1) the coordinates of the focus referred to some rectangular axes in the plane, and

$$Ax + By + C = 0$$

the equation of the directrix referred to these same axes.

Then if (x, y) be any point on the particular curve under consideration we have

$$(x - x_1)^2 + (y - y_1)^2 = e^2 \frac{(Ax + By + C)^2}{A^2 + B^2},$$

for the left-hand side of this equation is the square of the distance of (x, y) from the focus (x_1, y_1), and the right-hand side is e^2 times the square of the distance from the directrix.

It is clear that the above equation is of the second degree, being of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The form would be exactly the same if the Cartesian axes were inclined at an angle ω other than a right angle, for then we should have (§ 109)

$$(x-x_1)^2+(y-y_1)^2+2(x-x_1)(y-y_1)\cos\omega=e^2\frac{(Ax+By+C)^2\sin^2\omega}{(A^2+B^2-2AB\cos\omega)}.$$

113. We now see that every conic section is represented by an equation of the second degree. That this must be so is also clear from the fact that the conic sections being projections of a circle, which is such that every straight line in its plane meets it in two points (which may be imaginary or

coincident), must themselves have the same property, since the projection of a straight line is another straight line. Thus when we eliminate x between $Ax+By=C$ (the equation of a line) and the equation of a conic section, we must obtain a quadratic equation in y. This can only be the case if the equation of the conic section be of the second degree in x and y.

We shall presently go on to shew that an equation of the second degree always represents a conic section, that is to say it will represent two straight lines, or a circle, or, failing these, an ellipse or a parabola or a hyperbola. Before proving this, we shall obtain the equations of the parabola, ellipse and hyperbola in their simplest standard forms.

114. Tangent to a conic. Since a conic section, or 'conic' as we shall call it, is a curve of the second degree, every straight line in its plane will meet it in two points, real or imaginary (compare § 71). In special cases the two points in which a line meets a conic will be coincident, and then we call the line a *tangent* to the conic.

A tangent to a conic then is a line which meets it in two coincident points.

115. Standard form of the equation of the parabola. Let S be the focus, DX the directrix.

Draw SX perpendicular to the directrix and take first of all XS and XD for axes of coordinates. Let $XS=c$, so that the coordinates of S are $(c, 0)$.

Let P be any point on the parabola, (x, y) its coordinates.

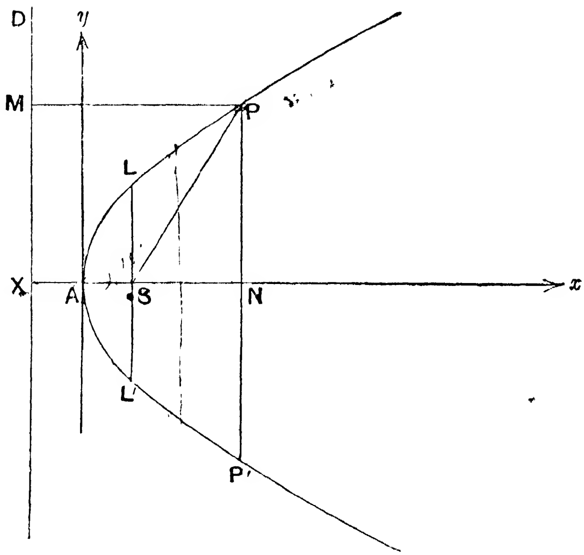
Draw PM perpendicular to the directrix.

Then
$$SP=PM,$$
$$\therefore\ SP^2=PM^2,$$
$$\therefore\ (x-c)^2+y^2=x^2$$

for the left side is the square of the distance of (x, y) from $(c, 0)$,

$$\therefore\ y^2 = 2cx - c^2$$

$$= 2c\left(x - \frac{c}{2}\right).$$



Put $y = 0$ in this to find the point A where the curve cuts the x-axis. We have

$$x = \frac{c}{2}.$$

If we now transfer the origin to $A\left(\frac{c}{2}, 0\right)$ leaving the axes unchanged in direction, our equation becomes

$$y^2 = 2cx.$$

It is usual to write a for XA or AS, so that $c = 2a$. The equation is then

$$y^2 = 4ax.$$

This is the standard equation of the parabola.

116. Some properties of the curve. The point A is called the *vertex*. The line AS produced indefinitely is called the *axis*.

We see that the parabola is symmetrical with regard to its axis. For to every point (x, y) on the curve there corresponds a point $(x, -y)$.

If PN be drawn perpendicular to the axis and produced to meet the curve in P', PN is often called the *ordinate* of the point P, and PNP' the *double ordinate* of P, while AN is called the *abscissa* of P, it being the portion of the axis cut off, as it were, by the ordinate.

The double ordinate LSL' which passes through the focus is called the *latus rectum* of the parabola.

Let $SL = l$.

Therefore the coordinates of L are (a, l).

But L is on the curve $y^2 = 4ax$,

$$\therefore \quad l^2 = 4a^2.$$

Thus we see that $SL = 2a$, that is the latus rectum is of length $4a$.

We now observe that the axis of y is a tangent to the parabola $y^2 = 4ax$.

For putting $x = 0$ in this equation we get $y^2 = 0$, that is

$$y = 0 \ \textit{bis}.$$

Thus the line $x = 0$ meets the curve in the two points $(0, 0)$, $(0, 0)$, that is to say, in two coincident points. Therefore it is a tangent to the parabola.

We see that negative values of x would give imaginary values of y, thus the curve lies wholly in the positive direction of the x-axis. Moreover as x can have any positive value, however great, we see that the curve extends to infinity.

117. We can now see that a curve whose equation referred to two rectangular axes is

$$y^2 = 4ax$$

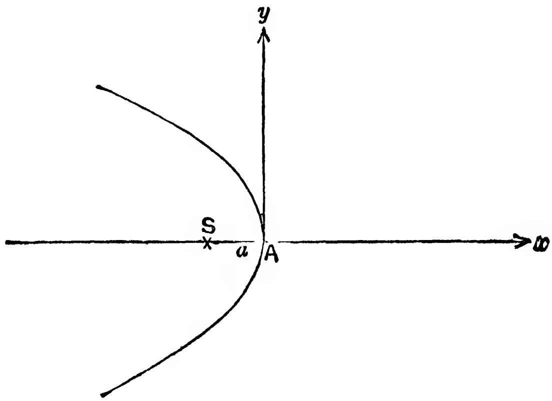
is a parabola, for on working the algebra of § 115 backwards we can prove that this curve is the locus of points whose distance from $(a, 0)$ is equal to their distance from the line $x = -a$.

Also we can see that the curve whose equation is

$$y^2 = -4ax$$

is a parabola, for, if we write $x' = -x$, the equation becomes

$$y^2 = 4ax'.$$



That is to say, $y^2 = -4ax$ is a parabola whose axis runs in the negative direction of the x-axis as in the figure.

Thus a curve which is the locus of points the square of whose distance from one line (l) varies as their distance from another line (l') perpendicular to the first is a parabola, having l for its axis and l' for the tangent at the vertex. And the constant of variation is the length of the latus rectum of the parabola.

Thus the equation

$$(Ax + By + C)^2 = k\,(Bx - Ay + C')$$

represents a parabola, whose axis has for equation

$$Ax + By + C = 0,$$

and the tangent at the vertex is the perpendicular line

$$Bx - Ay + C' = 0.$$

The above equation can be written

$$\left(\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right)^2=\frac{k}{\sqrt{A^2+B^2}}\,\frac{Bx-Ay+C'}{\sqrt{A^2+B^2}},$$

so that the length of the latus rectum is the numerical value of

$$\frac{k}{\sqrt{A^2+B^2}}.$$

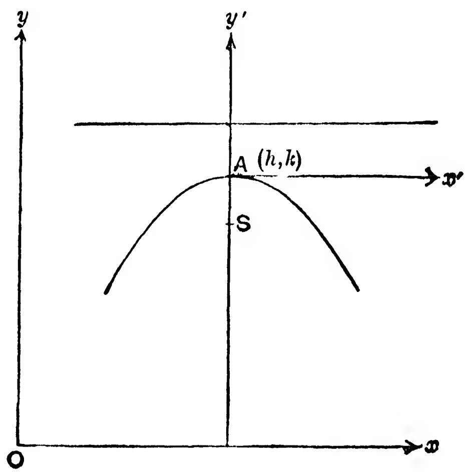
Examples. 1. Find the focus and directrix of the parabola

$$(x-h)^2+4a(y-k)=0.$$

[If we transfer the origin to (h, k) the equation becomes

$$X^2=-4aY,$$

which is a parabola of latus rectum $4a$ with its vertex at the new origin and with its axis running in the negative direction of the Y-axis.



Hence the vertex of the parabola referred to the original axes is (h, k).
The x-coordinate of the focus is h.
The y-coordinate of the focus is $k-a$.
The directrix is $y=k+a$.]

2. Find the vertex, focus and directrix of the parabola

$$y^2+4x+2y-8=0$$

and represent the same in a figure.

[We write this equation

$$(y^2+2y+1)+4x-9=0,$$

that is
$$(y+1)^2=-4\left(x-\tfrac{9}{4}\right).$$

The vertex is $(\frac{9}{4}, -1)$, the focus $(\frac{5}{4}, -1)$, and the directrix has for its equation $4x-13=0$.]

3. Find the equation of the parabola whose focus is the point $(\frac{5}{4}, -1)$ and whose directrix is the line $4x-13=0$.

4. Find the length of the latus rectum and the position of the vertex, focus, and directrix of the following parabolas:

(i) $(y-3)^2+2(x-2)=0$.

(ii) $(x-2)^2=5(y+1)$.

(iii) $y^2+2x-4y+3=0$.

(iv) $x^2+4x-3y=0$.

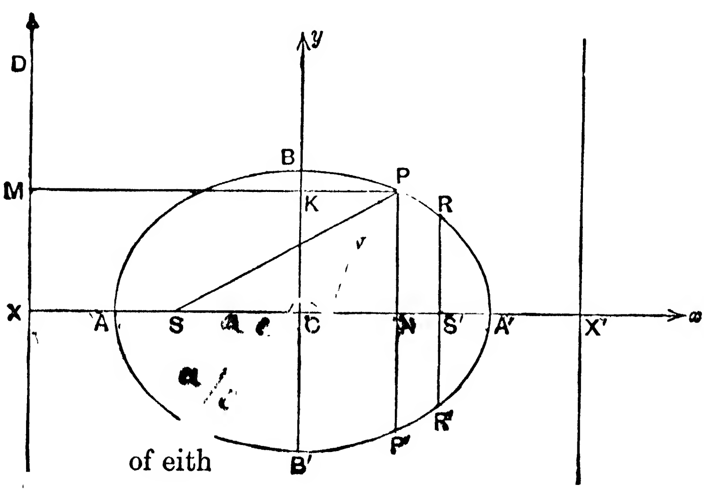
(v) $y^2+2gx+2fy+c=0$.

5. Find the equations of the two parabolas whose latus rectum is 6 and the axis and tangent at the vertex are the lines whose equations are

$$3x+4y+1=0, \quad 4x-3y=0.$$

118. Standard form of the equation of the ellipse. Let S be the focus, DX the directrix, e the eccentricity, which is necessarily less than unity.

Draw SX perpendicular to the directrix and as before take first XS and XD for axes of coordinates. Let $XS=c$.



Let P be any point on the ellipse, (x, y) its coordinates. Draw PM perpendicular to the directrix.

$$\therefore \; SP = e \,.\, PM,$$

$$\therefore \; SP^2 = e^2 \,.\, PM^2,$$

$$\therefore \; (x-c)^2 + y^2 = e^2 \,.\, x^2,$$

$$\therefore \; x^2(1-e^2) - 2cx + y^2 + c^2 = 0,$$

$$\therefore \; x^2 - \frac{2c}{1-e^2}x + \frac{y^2}{1-e^2} = -\frac{c^2}{1-e^2},$$

$$\therefore \; \left(x - \frac{c}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{c^2}{(1-e^2)^2} - \frac{c^2}{1-e^2} = \frac{e^2c^2}{(1-e^2)^2}.$$

Now transfer the origin to the point C whose coordinates are $\left(\frac{c}{1-e^2}, 0\right)$ and write $\frac{ec}{1-e^2} = a$.

The equation becomes

$$x^2 + \frac{y^2}{1-e^2} = a^2,$$

that is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(1-e^2)$, a quantity necessarily positive since $e < 1$.

This is the equation of the ellipse in its standard form.

119. Some properties of the curve. The curve is symmetrical with regard to both the x-axis and the y-axis. For if (x, y) be a point on the curve, so also are $(-x, y)$, $(x, -y)$ and $(-x, -y)$. Every line through C will thus meet the curve in two points equidistant from C. That is, every chord passing through C is bisected at C. The point C is therefore called the *centre*.

Putting $y = 0$ in the equation of the curve we find $x = \pm a$. Thus the two points A and A' in which the curve cuts the x-axis are distant a from the centre.

Putting $x = 0$ we get $y = \pm b$. Thu[illegible] [illegible]oints B and B' in which the curve cuts the y-axis are dist[illegible] [illegible] the [illegible]re.

If $x > a$, y is imaginary, and if $y > b$, x is imaginary. Thus the curve is limited and closed.

The two lines AA' and BB' with respect to which the curve is, as we have said, symmetrical are called the *axes* of the ellipse. As $AA' > BB'$ (for $b^2 = a^2(1 - e^2)$ so that $b < a$) AA' is called the *major axis*, and BB' the *minor axis*.

We shall now prove that $CS = e . CA$ and $CA = e . CX$.

Since A and A' are points on the curve,

$$AS = e . XA,$$

$$SA' = e . XA'.$$

Adding, we have

$$AA' = e(XA + XA') = 2e . XC,$$

$$\therefore \; CA = e . CX.$$

Subtracting, we get

$$SA' - AS = e . AA',$$

$$\therefore \; SC + CA' - (AC - SC) = e . AA',$$

$$\therefore \; CS = e . CA.$$

The symmetry of the curve exhibited by its equation shews that there must be a second focus S' situated on the major axis at the same distance from C as S, and a second directrix corresponding with S' and parallel to the original directrix and cutting the major axis produced in X' where $CX' = XC$.

If from a point P on the curve PN be drawn perpendicular to the major axis, and produced to meet the curve again in P', PN is called the *ordinate* and PNP' the *double ordinate* of the point P. A double ordinate through a focus is called a *latus rectum*.

The length of either latus rectum is $\dfrac{2b^2}{a}$.

For let $RS'R'$ be the latus rectum through S' and let $S'R=l$; then the coordinates of R are (ae, l). But R is on the curve,

$$\therefore \frac{a^2e^2}{a^2}+\frac{l^2}{b^2}=1,$$

$$\therefore \frac{l^2}{b^2}=(1-e^2)=\frac{b^2}{a^2},$$

$$\therefore l^2=\frac{b^4}{a^2},$$

$\therefore$ the length of the semi-latus rectum is $\frac{b^2}{a}$.

120. Geometrical property expressed by the standard equation of the ellipse.

If PN be the ordinate of the point P, then from the equation of the ellipse we have

$$\frac{CN^2}{a^2}+\frac{PN^2}{b^2}=1,$$

$$\therefore \frac{PN^2}{b^2}=1-\frac{CN^2}{a^2}=\frac{CA^2-CN^2}{a^2}=\frac{AN.NA'}{a^2},$$

$$\therefore \frac{PN^2}{AN.NA'}=\frac{b^2}{a^2}=\frac{BC^2}{AC^2}.$$

This is a geometrical property of the ellipse probably already familiar to the student.

In the same way, if PK be drawn perpendicular to the minor axis,

$$\frac{PK^2}{BK.KB'}=\frac{AC^2}{BC^2},$$

for we have $$\frac{PK^2}{a^2}+\frac{CK^2}{b^2}=1.$$

121. The circle as a limiting case of the ellipse. We see that if e becomes smaller while a remains constant ae becomes smaller and the foci approach the centre; also b^2 which $=a^2(1-e^2)$ approximates to a^2. Thus as the foci approach

the centre, the ellipse tends to become more circular in appearance, and when e becomes very small and nearly equal to zero the ellipse becomes almost a circle; so we say that a circle of radius a is the limiting case of an ellipse whose major axis is $2a$ and whose eccentricity tends to zero.

122. The parabola as a limiting case of the ellipse. If we take the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and transfer the origin to the vertex A whose coordinates are $(-a, 0)$ the equation becomes

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1,$$

that is $$\frac{x^2}{a^2}-\frac{2x}{a}+\frac{y^2}{b^2}=0,$$

that is $$\frac{x^2}{a}-2x+\frac{y^2}{a(1-e^2)}=0.$$

Now $AS=a(1-e)$; denote this by d,

$$\therefore\ a=\frac{d}{1-e}.$$

Thus the equation is

$$\frac{x^2(1-e)}{d}-2x+\frac{y^2}{d(1+e)}=0.$$

Now let us suppose that d remains finite while $1-e$ becomes very small, then a becomes very large and the equation of the curve approximates to $y^2=4dx$, which is a parabola. Thus a parabola may be regarded as the limiting case of an ellipse when the centre moves off to a great distance, the vertex A and the focus S remaining unchanged.

123. We see from the equation of the ellipse that if we have two intersecting perpendicular lines l and l' and a point P moves in their plane so that

$$\frac{Pl^2}{a^2}+\frac{Pl'^2}{b^2}=1,$$

where p_l and $p_{l'}$ are the lengths of the perpendiculars from P on l and l' respectively and a and b are real quantities, so that a^2 and b^2 are positive, then the point P will describe an ellipse having its centre at the intersection of the two lines, and, if a be $> b$, having its major axis, of length $2a$, lying along the line l' and its minor axis, of length $2b$, lying along the line l.

Examples. 1. Find the lengths of the axes and of the latera recta of the ellipse $4x^2+3y^2=24$.

[We write this
$$\frac{x^2}{6}+\frac{y^2}{8}=1.$$

Thus in this case the axis of x lies along the minor axis and that of y along the major axis. We have $a^2=8$, $b^2=6$, therefore the lengths of the axes are $2\sqrt{8}$ $(=4\sqrt{2})$ and $2\sqrt{6}$ and the latera recta are of length
$$\frac{2b^2}{a}=\frac{2\times 6}{2\sqrt{2}}=3\sqrt{2}.]$$

2. Find the centre and eccentricity of the ellipse
$$2x^2+3y^2-4x+5y+4=0.$$

[We write this equation
$$2(x^2-2x+1)+3(y^2+\tfrac{5}{3}y+\tfrac{25}{36})=2+\tfrac{25}{12}-4=\tfrac{1}{12},$$

that is
$$\frac{(x-1)^2}{\frac{1}{24}}+\frac{(y+\frac{5}{6})^2}{\frac{1}{36}}=1.$$

If we transfer the origin to $(1, -\frac{5}{6})$ the equation becomes $\dfrac{X^2}{\frac{1}{24}}+\dfrac{Y^2}{\frac{1}{36}}=1.$

Thus the equation represents an ellipse whose centre is at $(1, -\frac{5}{6})$, and if a and b be the semi-major and minor axes $a^2=\frac{1}{24}$, $b^2=\frac{1}{36}$.
$$\therefore\quad \tfrac{1}{36}=\tfrac{1}{24}(1-e^2),$$
$$\therefore\quad 1-e^2=\tfrac{2}{3},\qquad \therefore\quad e=\frac{1}{\sqrt{3}}.]$$

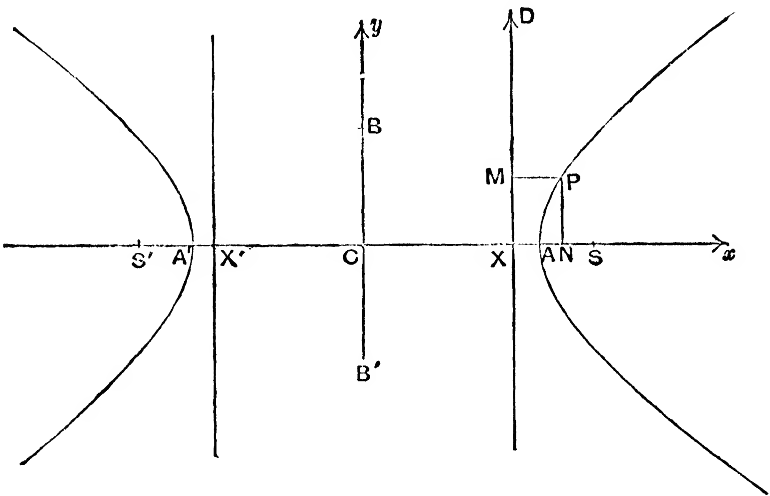
3. Find the coordinates of the foci of the ellipse of Ex. 2.

4. Find the equation of an ellipse whose axes are of lengths 6 and 8 and their equations $3x-4y+1=0$, $4x+3y-2=0$ respectively.

124. Standard form of the equation of the hyperbola.

The work for obtaining the equation of the hyperbola in its standard form is very similar to that already done for the ellipse. There are some important points of difference however.

Let S be the focus, DX the directrix, and e the eccentricity, which is > 1.



Draw SX perpendicular to the directrix and first take XS and XD for axes of coordinates. Let $XS = c$.

Let (x, y) be the coordinates of any point P on the curve Draw PM perpendicular to the directrix,

$$\therefore \; SP^2 = e^2 . PM^2,$$

$$\therefore \; (x-c)^2 + y^2 = e^2x^2,$$

$$\therefore \; x^2(e^2-1) + 2cx - y^2 = c^2,$$

$$\therefore \; x^2 + \frac{2c}{e^2-1}x - \frac{y^2}{e^2-1} = \frac{c^2}{e^2-1},$$

$$\therefore \; \left(x + \frac{c}{e^2-1}\right)^2 - \frac{y^2}{e^2-1} = \frac{c^2}{(e^2-1)^2} + \frac{c^2}{e^2-1} = \frac{c^2e^2}{(e^2-1)^2}.$$

Now transfer the origin to the point C whose coordinates are $\left(-\frac{c}{e^2-1}, 0\right)$ and write $\frac{ce}{e^2-1} = a$.

The equation is now

$$x^2 - \frac{y^2}{e^2-1} = a^2,$$

that is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(e^2-1)$, a quantity necessarily positive since $e > 1$.

This is the standard form of the equation of the hyperbola.

125. Some properties of the curve. If (x, y) be any point on the curve, then $(-x, y)$, $(x, -y)$, $(-x, -y)$ also lie on the curve, which is therefore symmetrical about both axes. Lines through C will meet the curve in two points equidistant from C. That is chords of the curve which pass through C will be bisected at C. This point then is called the *centre.*

Putting $y=0$ in the equation of the curve we find $x = \pm a$. Thus the two points A and A' in which the curve cuts the x-axis are distant a from the centre.

Putting $x=0$ we get $y^2 = -b^2$; thus the curve does not meet the y-axis in any *real* point, but in two imaginary points distant $b\sqrt{-1}$ from C.

Any value of x lying between $-a$ and $+a$ would make y^2 negative so that no part of the curve lies between A and A'. x can however have any positive or negative value numerically greater than a. The curve then extends to infinity in both directions, and consists of two branches.

The line AA' is called the *transverse axis.* If on the y-axis we take two points B and B' each distant b from C, then BB' is called the *conjugate axis.* But it must be carefully observed that B and B' are not points on the curve.

It is easy to prove that $CS = e.CA$ and $CA = e.CX$ as in § 119.

For $$AS = e \,.\, XA,$$
$$A'S = e \,.\, A'X,$$

adding we get $2CS = 2e \,.\, CA$, that is $CS = e \,.\, CA$.

Subtracting we get $A'A = e\,(A'C + CX) - e\,(CA - CX)$, that is $CA = e \,.\, CX$.

As in the case of the ellipse, there must be a second focus S', situated on the transverse axis at the same distance from the centre as S, and a directrix to correspond with S'.

PN drawn perpendicular to the transverse axis is called the *ordinate* of the point P, and if PN be produced to meet the curve again in P', PNP' is called a *double ordinate*. A double ordinate through a focus is called a *latus rectum*. It is easy to prove, as in § 119, that each latus rectum is of length $\dfrac{2b^2}{a}$.

It can be shewn further, as in § 120, that the Geometrical property of the hyperbola expressed by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is that which is usually written $\dfrac{PN^2}{AN \,.\, A'N} = \dfrac{BC^2}{AC^2}$.

We can see too (compare § 123) that if we have two intersecting lines l and l', and a point P moves in their plane so that

$$\frac{p_l^2}{a^2} - \frac{p_{l'}^2}{b^2} = 1,$$

where p_l and $p_{l'}$ are the perpendiculars from P on l and l', and a and b are real quantities so that a^2 and b^2 are both positive, then the locus of P is a hyperbola, the length of whose transverse axis, lying along l', is $2a$, and the length of its conjugate axis, lying along l, is $2b$.

126. Two straight lines as the limiting case of a hyperbola.

The equation of a hyperbola referred to its axes being

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1,$$

which we can write
$$x^2 - \frac{y^2}{e^2-1} = a^2,$$

we see that if e be kept constant while a is made gradually smaller and smaller until it becomes very small indeed the hyperbola will approximate to the two straight lines

$$x^2 - \frac{y^2}{e^2-1} = 0,$$

the axes of coordinates being the bisectors of the angles between them.

Thus we may regard a pair of straight lines as the limiting case of a hyperbola whose axes are infinitely small, while their ratio is finite.

127. Rectangular hyperbola. In the special case in which $b=a$ the hyperbola is called *rectangular*. This name is explained by the fact that when $b=a$ the 'asymptotes,' of which we shall speak in a later chapter, are at right angles. The name 'equilateral' has also been applied to such hyperbolas.

Examples. 1. Find the centre and the length of the transverse axis of the hyperbola

$$x^2-2y^2-2x+8y-1=0.$$

[We write this
$$x^2-2x-2(y^2-4y)=1,$$
i.e.
$$(x-1)^2-2(y-2)^2=1+1-8=-6,$$
i.e.
$$\frac{(y-2)^2}{3} - \frac{(x-1)^2}{6} = 1.$$

Thus the centre is at ~~(1, 2)~~, and the transverse axis which is parallel to the y-axis is of length $2\sqrt{3}$.] → How.

2. Find the lengths of the axes, and the eccentricity of the hyperbola

$$x^2-3y^2-2x=8.$$

3. Find the equation of the hyperbola the lengths of whose transverse and conjugate axes are respectively 4 and 6, the equations of these axes being respectively $3x+4y-1=0$ and $4x-3y+2=0$.

128. The General Equation of the second degree.

PROP. *Every Cartesian equation of the second degree represents a conic.*

For consider the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{.........(1)}.$$

First suppose the axes of coordinates are rectangular.

Turn the axes through an angle θ and choose θ so that the xy term of the new equation vanishes (§ 107).

Our new equation will be (say)

$$a'x'^2+b'y'^2+2g'x'+2f'y'+c=0 \quad \text{............(2)},$$

so that $$ax^2+2hxy+by^2=a'x'^2+b'y'^2.$$

From the theory of invariants (§ 106) we have

$$a+b=a'+b',$$
$$ab-h^2=a'b'.$$

(1) If $ab=h^2$ either a' or b' is zero. Suppose a' is zero, the equation is then

$$b'y'^2+2g'x+2f'y+c=0,$$

which is a parabola having its axis parallel to the x-axis.

(2) If $ab \neq h^2$, then neither a' nor b' can be zero and our equation can be written

$$a'\left(x'^2+\frac{2g'}{a'}x'\right)+b'\left(y'^2+\frac{2f'}{b'}y'\right)+c=0.$$

Complete the squares of the terms in x' and y' and this becomes

$$a'\left(x'+\frac{g'}{a'}\right)^2+b'\left(y'+\frac{f'}{b'}\right)^2=\frac{g'^2}{a}+\frac{f'^2}{b}-c,$$

which represents (i) an ellipse with its centre at $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ if a' and b' have the same sign, (ii) a hyperbola with its centre

at $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ if a' and b' have different signs, and if in particular $a'+b'=0$ the hyperbola will be a rectangular one (§ 127).

Now a' and b' will have the same or opposite signs according as $a'b'$ is positive or negative, that is, according as $ab-h^2$ is positive or negative.

Thus if $ab-h^2$ is positive (1) will represent an ellipse, but if $ab-h^2$ is negative (1) will represent a hyperbola, and if $a'+b'=0$, and therefore also $a+b=0$, the hyperbola will be a rectangular one.

Next let the axes of coordinates be inclined at an angle ω. Transform to rectangular axes with the same origin so that equation (1) will transform to (say)

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x + 2f'y + c = 0.$$

This will represent

(i) a parabola if $a'b' - h'^2 = 0$,

(ii) an ellipse if $a'b' - h'^2 > 0$,

(iii) a hyperbola if $a'b' - h'^2 < 0$,

(iv) a rectangular hyperbola if $a' + b' = 0$.

But by invariants we have (§ 106)

$$\frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{a'+b'-2h'\cos\frac{\pi}{2}}{\sin^2\frac{\pi}{2}} = a'+b',$$

$$\frac{ab-h^2}{\sin^2\omega} = \frac{a'b'-h'^2}{\sin^2\frac{\pi}{2}} = a'b'-h'^2.$$

Thus

(i) if $ab - h^2 = 0$ (1) is a parabola,

(ii) if $ab - h^2 > 0$ (1) is an ellipse,

(iii) if $ab - h^2 < 0$ (1) is a hyperbola,

(iv) if $a + b - 2h\cos\omega = 0$ (1) is a rectangular hyperbola.

It will be observed that the condition that the equation (1) should represent a parabola is that the terms of the highest degree, viz. $ax^2 + 2hxy + by^2$ should form a perfect square.

129. Summary. The results to be remembered then are that the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will be (i) two straight lines if (§ 55)

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0,$$

(ii) a circle if $a = b$, and $h = 0$, the axes being rectangular; or if $a : b : h = 1 : 1 : \cos\omega$, the axes being oblique,

(iii) an ellipse if $ab - h^2 > 0$, and the conditions for a circle be not satisfied,

(iv) a hyperbola if $ab - h^2 < 0$, and the condition for two straight lines be not satisfied,

(v) a rectangular hyperbola if $a + b - 2h\cos\omega = 0$, and the condition for two straight lines be not satisfied.

If both the condition for two straight lines and the relation $a + b - 2h\cos\omega = 0$ be satisfied, the equation represents two straight lines at right angles (§ 110).

130. We shall go on in the following chapters to evolve the properties of the various conics by considering them to be expressed in their standard forms. It will greatly simplify our work if before passing on to this we obtain certain equations which are applicable to all the conics alike, and which are really as easy in the general case as they are in the particular ones. The student is recommended to make the contents of the present chapter his own before passing on. The rest of the subject will be found to be thus greatly simplified.

All the results that we shall obtain in the rest of this chapter hold for oblique, quite as well as for rectangular axes.

131. The Tangent.

T = 0

To find the equation of the tangent to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots(1)$$

at the point (x_1, y_1).

The equation of a line through (x_1, y_1) is as we have seen (§ 32)

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where l and m are constants depending only on the direction of the line and r is the algebraical distance of (x_1, y_1) from (x, y).

To find where (2) meets the conic substitute

$$x = x_1 + lr, \quad y = y_1 + mr$$

into the equation of the conic; we thus get

$$a(x_1 + lr)^2 + 2h(x_1 + lr)(y_1 + mr) + b(y_1 + mr)^2$$
$$+ 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0,$$

which gives

$$(al^2 + 2hlm + bm^2)r^2$$
$$+ 2r\{(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m\}$$
$$+ ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \ldots(3).$$

Now if (2) be a tangent to the conic at (x_1, y_1), *both* of the roots of this quadratic equation in r must be zero, otherwise the line (2) will meet the conic in another point other than (x_1, y_1).

We thus have the conditions

$$\text{(i)} \quad ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0,$$

which is satisfied since (x_1, y_1) lies on the conic,

$$\text{(ii)} \quad (ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0,$$

this gives the ratio of $l : m$.

If now we eliminate this ratio $l : m$ from this relation and (2) we shall have the equation of the tangent, viz.:

$$\frac{x - x_1}{l} \times (ax_1 + hy_1 + g)l = -\frac{y - y_1}{m}(hx_1 + by_1 + f)m = 0,$$

that is

$$(x - x_1)(ax_1 + hy_1 + g) + (y - y_1)(hx_1 + by_1 + f) = 0.$$

This is the equation of the tangent at (x_1, y_1), but this is not the standard form.

We can reduce as follows:

$$\begin{aligned} axx_1 + h(xy_1 + x_1y) + byy_1 + gx + fy \\ &= ax_1^2 + 2hx_1y_1 + by_1^2 + gx_1 + fy_1 \\ &= -gx_1 - fy_1 - c, \end{aligned}$$

that is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

which is the standard form of the equation of the tangent.

It is convenient to write T for the expression on the left side.

132. On the form of the equation of the tangent. It is very important that the student should be able to write down with facility the equation of the tangent to any curve of the second degree as it arises. It is quite easy to do this if the following rules be remembered:

(1) In the terms in x^2 and y^2 in the equation of the conic write xx_1 and yy_1 respectively for x^2 and y^2.

(2) In the term in xy write $xy_1 + x_1y$ for $2xy$.

(3) In the terms in x and y write $x + x_1$ and $y + y_1$ for $2x$ and $2y$ respectively.

(4) Retain the constant term.

Observing these rules it will be seen that the tangent at (x_1, y_1) to

(i) $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$,

(ii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$,

(iii) $xy = k^2$ is $\frac{1}{2}(xy_1 + x_1y) = k^2$.

133. *The condition that the line*

$$lx + my + n = 0$$

should be a tangent to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is
$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & o \end{vmatrix} = 0.$$

For suppose the line touches the conic at (x_1, y_1), then it must be identical with

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

that is with

$$(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + (gx_1 + fy_1 + c) = 0,$$

$$\therefore \frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n} = \lambda \text{ (say)},$$

$$\therefore \quad ax_1 + hy_1 + g - l\lambda = 0,$$
$$hx_1 + by_1 + f - m\lambda = 0,$$
$$gx_1 + fy_1 + c - n\lambda = 0.$$

Also since (x_1, y_1) is on the line

$$lx_1 + my_1 + n = 0.$$

Eliminating x_1, y_1 and λ we get

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & o \end{vmatrix} = 0.$$

This multiplied out becomes

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

where A, B, C etc. are the 'prepared minors,' that is the minors taken with their proper sign of a, b, c, etc. in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Thus $\quad A = bc - f^2, \quad B = ca - g^2, \quad C = ab - h^2,$

$$F = gh - af, \quad G = hf - bg, \quad H = fg - ch.$$

134. We can see that *a pair of tangents can be drawn from a point not on a conic to the conic.*

Let the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let (x_1, y_1) be a point not on the conic. Suppose a line drawn through this point to touch the conic, and let (x_2, y_2) be the point of contact.

The equation of the tangent at (x_2, y_2) is

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

As (x_1, y_1) lies on this we have

$$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \ldots (1).$$

And as (x_2, y_2) is on the conic we have

$$ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \ldots\ldots (2).$$

These two equations (1) and (2) determine x_2 and y_2.

As (1) is a *simple* equation in x_2 and y_2, by substituting the value of x_2 in terms of y_2 into (2) we shall get a quadratic equation in y_2 which will in general have two roots. Thus there will be two possible points of contact of tangents from (x_1, y_1), but they will not be real in all cases.

135. *If a pair of tangents be drawn from* (x_1, y_1) *to the conic*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the equation of the 'chord of contact' (that is of the line through the points of contact) is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

For let (x_2, y_2) and (x_3, y_3) be the two points of contact of the tangents.

The equation of the tangent at (x_2, y_2) is

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

But (x_1, y_1) lies on this,

$$\therefore\ ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

Similarly as (x_1, y_1) lies on the tangent at (x_3, y_3)

$$ax_1x_3 + h(x_1y_3 + x_3y_1) + by_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0.$$

These two relations shew that (x_2, y_2) and (x_3, y_3) lie on

$$ax_1x + h(x_1y + xy_1) + by_1y + g(x_1 + x) + f(y_1 + y) + c = 0,$$

which represents a line as it is of the first order in x and y.

It is therefore the equation of the chord of contact required.

Hence the equation of the chord of contact of tangents from (x_1, y_1) *when* (x_1, y_1) *is not on the curve is exactly the same form as that of the equation of the tangent at* (x_1, y_1) *when* (x_1, y_1) *is on the curve.*

136. Poles and Polars.

We shall define *the* **polar** *of a point with respect to a conic to be the locus of the points of intersection of tangents at the extremities of chords through that point, and the point itself is called the* **pole** *of its polar.*

We can now prove that *the polar of* (x_1, y_1) *with respect to the conic*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $\quad axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

For let any chord be drawn through (x_1, y_1) and let the tangents at its extremities meet in (x_2, y_2), which is therefore a point on the polar of (x_1, y_1).

The chord of contact of tangents from (x_2, y_2) is (§ 135)

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0.$$

But (x_1, y_1) lies on this,

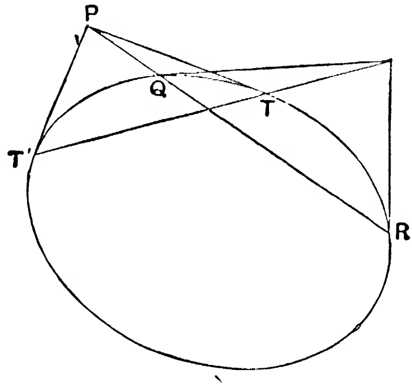
$$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

This relation shews that the locus of (x_2, y_2) is the line

$$ax_1x + h(x_1y + xy_1) + by_1y + g(x_1 + x) + f(y_1 + y) + c = 0.$$

This is the required equation, and we see from this that *the polar of a point with respect to a conic coincides with the chord of contact of tangents real or imaginary from that point to the conic.*

Thus if P be a point from which tangents PT and PT' are drawn to a conic, and PQR be any chord through P, the tangents at Q and R meet on the line TT'.



137. Conjugate points and lines.

If the polar of P passes through Q then the polar of Q will pass through P.

For if (x_1, y_1) be the point P and (x_2, y_2) the point Q, and the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the polar of P is

$$axx_1 + h(x_1y + xy_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

As (x_2, y_2) lies on this

$$ax_2x_1 + h(x_1y_2 + x_2y_1) + by_2y_1 + g(x_2 + x_1) + f(y_2 + y_1) + c = 0,$$

which is easily seen to be the condition that the polar of (x_2, y_2) should pass through (x_1, y_1).

Two points such that each lies on the polar of the other are called 'conjugate points.'

The condition that (x_1, y_1) and (x_2, y_2) should be conjugate points is then

$$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0.$$

Again we can prove that *if the pole of a line l lies on another line l', then the pole of l' lies on l.*

For let the pole of the line l be (x_1, y_1) and the pole of l' be (x_2, y_2), therefore the equation of the line l is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \ldots (1),$$

and the equation of l' is

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0 \ldots (2).$$

Now by hypothesis (x_1, y_1) lies on (2)

$$\therefore \; ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0,$$

therefore (x_2, y_2) lies on (1).

Thus the proposition is proved.

Two such lines are called *conjugate lines.*

When three points A, B, C are such that every two of them are conjugate points, then the triangle ABC is said to be a 'self-conjugate (or self-polar) triangle.' Since the polar of A passes through both B and C, BC is the polar of A. So that each side of the triangle is the polar of the opposite vertex. And any two sides of the triangle are seen to be conjugate lines.

138. *The condition that the lines*

$$l_1x + m_1y + n_1 = 0 \quad \ldots\ldots\ldots\ldots (1),$$

$$l_2x + m_2y + n_2 = 0 \quad \ldots\ldots\ldots\ldots (2),$$

should be conjugate lines for the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$\begin{vmatrix} a, & h, & g, & l_1 \\ h, & b, & f, & m_1 \\ g, & f, & c, & n_1 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = 0.$$

For let (x_1, y_1) be the pole of (1), therefore (1) is identical with

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0,$$

that is with

$$(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + (gx_1 + fy_1 + c) = 0,$$

$$\therefore \frac{ax_1+hy_1+g}{l_1} = \frac{hx_1+by_1+f}{m_1} = \frac{gx_1+fy_1+c}{n_1} = \lambda \text{ (say)},$$

$$\therefore \; ax_1+hy_1+g-l_1\lambda=0,$$
$$hx_1+by_1+f-m_1\lambda=0,$$
$$gx_1+fy_1+c-n_1\lambda=0,$$

and since (x_1, y_1) by hypothesis lies on (2)

$$l_2x_1+m_2y_1+n_2=0.$$

Eliminating x_1, y_1 and λ from these equations we get

$$\begin{vmatrix} a, & h, & g, & l_1 \\ h, & b, & f, & m_1 \\ g, & f, & c, & n_1 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = 0.$$

139. The Centre.

If (x_1, y_1) *be the centre of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

then x_1 *and* y_1 *are given by the equations*

$$ax_1+hy_1+g=0.$$
$$hx_1+by_1+f=0.$$

For a line through (x_1, y_1) can be written

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = r.$$

Where this meets the conic (§ 131)

$$(al^2+2hlm+bm^2)\,r^2+2r\,\{(ax_1+hy_1+g)\,l+(hx_1+by_1+f)\,m\}$$
$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0.$$

Now if (x_1, y_1) be the centre, *every* line through (x_1, y_1) meets the curve in two points equidistant from (x_1, y_1), that is the roots of this quadratic equation r are equal in magnitude but opposite in sign for all values of l and m. Thus the coefficient of r must vanish for all values of l and m, that is

$$ax_1+hy_1+g=0,$$
$$hx_1+by_1+f=0.$$

The student will remember that these were the equations giving the point of intersection of the lines represented by the general equation in the case where it represents two straight lines.

140. The result of the previous article could also be obtained in the following way:

If we transfer the origin to (x_1, y_1) by writing

$$x = X + x_1, \quad y = Y + y_1,$$

the equation of the conic becomes

$$aX^2 + 2hXY + bY^2 + 2(ax_1 + hy_1 + g)X + 2(hx_1 + by_1 + f)Y$$
$$+ ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

But, the origin being now at the centre, the terms in X and Y of the first order must disappear,

$$\therefore \quad ax_1 + hy_1 + g = 0,$$
$$hx_1 + by_1 + f = 0.$$

From these we find

$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{gh - af}{ab - h^2},$$

that is, using the notation of § 133,

$$x_1 = \frac{G}{C}, \quad y_1 = \frac{F}{C}.$$

141. Equation of a chord in terms of its middle point.

If (x_1, y_1) be the middle point of a chord of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the equation of the line of the chord is

$$T = S_1,$$

where

$$T \equiv axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c,$$

and S_1 is what S becomes when x_1 and y_1 are written for x and y.

Let $$\frac{x-x_1}{l}=\frac{y-y_1}{m}=r \quad \text{.......................(1)}$$

be the equation of the line of the chord.

As before we write x_1+lr, y_1+mr for x and y in the equation of the conic and get

$$(al^2+2hlm+bm^2)\,r^2+2r\,\{(ax_1+hy_1+g)\,l+(hx_1+by_1+f)\,m\}$$
$$+(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)=0.$$

But as (x_1, y_1) is the middle point of the chord the values of r furnished by this equation must be equal in magnitude and opposite in sign,

$$\therefore\ (ax_1+hy_1+g)\,l+(hx_1+by_1+f)\,m=0.$$

Eliminating the ratio of $l:m$ between this and (1) we get

$$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0$$

as the equation of the chord.

This can be written

$$axx_1+h(xy_1+x_1y)+byy_1+gx+fy$$
$$=ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1.$$

Adding gx_1+fy_1+c to both sides we get

$$T=S_1.$$

COR. *The locus of the middle points of a series of parallel chords is a line through the centre.*

For if the chords be parallel to the line $Ax+By=0$ we have, if (x_1, y_1) be the middle point of one of the chords

$$\frac{ax_1+hy_1+g}{A}=\frac{hx_1+by_1+f}{B}.$$

Thus the locus of (x_1, y_1) is the line

$$\frac{ax+hy+g}{A}=\frac{hx+by+f}{B},$$

which is satisfied by $\left.\begin{aligned}ax+hy+g&=0\\hx+by+f&=0\end{aligned}\right\}$

that is by the centre.

142. Equation of pair of tangents from a point.

The equation of the pair of tangents to the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

from the point (x_1, y_1) *is* $\quad SS_1 = T^2.$

For the equation of a line through (x_1, y_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \quad \text{.....................(1).}$$

To find where this meets the conic we have the quadratic in r

$$(al^2 + 2hlm + bm^2)\, r^2 + 2r\,\{(ax_1 + hy_1 + g)\, l + (hx_1 + by_1 + f)\, m\} + S_1 = 0.$$

Now if (1) be a tangent the roots of this equation in r must be equal, that is

$$S_1 (al^2 + 2hlm + bm^2) = \{(ax_1 + hy_1 + g)\, l + (hx_1 + by_1 + f)\, m\}^2.$$

Thus eliminating the ratio of $l : m$ between this and (1) we see that any point on a tangent from (x_1, y_1) to the conic must satisfy the equation

$$S_1 \{a (x - x_1)^2 + 2h (x - x_1)(y - y_1) + b (y - y_1)^2\}$$
$$= \{(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1)\}^2.$$

It will be found that this can be written

$$S_1 \{S + S_1 - 2T\} = (T - S_1)^2,$$

which gives $\quad SS_1 = T^2.$

This then being satisfied by all points on either tangent from (x_1, y_1) must be the equation of the pair of tangents.

Another method of getting this same result will be given in a later chapter.

143. Retrospect. On looking back over this chapter the student will see how extremely easy the results that we have obtained are to remember. It will be necessary to remember the standard forms of the equations of the parabola, ellipse and hyperbola, and to know the meaning of the constants involved

CHAPTER X.

THE HYPERBOLA.

181. The equation of the hyperbola referred to its 'principal axes' has been obtained in § 124, viz.,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

We have then the following equations:

Equation of the tangent, at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Equation of chord of contact of tangents from (x_1, y_1) and of polar of (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Equation of chord whose middle point is (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}.$$

Equation of pair of tangents from (x_1, y_1) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2.$$

Exactly as in the case of the ellipse we get the condition that $y = mx + c$ should touch the hyperbola, viz.,

$$c^2 = m^2a^2 - b^2,$$

and the locus of the points of intersection of tangents at right angles is the circle

$$x^2 + y^2 = a^2 - b^2,$$

known as the 'director circle.'

182. Coordinates expressed in terms of a single parameter.

To correspond with the representation of a point on ellipse by means of the eccentric angle, the coordinates of a point (x, y) on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

may be expressed

$$x = a \sec \theta, \quad y = b \tan \theta.$$

By giving θ different values all possible positions of (x, y) on the curve will be allowed for.

Or instead of using $\sec \theta$ and $\tan \theta$, we may use 'hyperbolic sines and cosines' and write

$$x = a \cosh u, \quad y = b \sinh u,$$

where
$$\cosh u = \frac{e^u + e^{-u}}{2},$$

and
$$\sinh u = \frac{e^u - e^{-u}}{2},$$

so that
$$\cosh^2 u - \sinh^2 u = 1.$$

Taking $(a \sec \theta, b \tan \theta)$ as the coordinates of a point P on the hyperbola we see that the equation of the tangent at P is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1,$$

for this comes at once by writing

$$x_1 = a \sec \theta, \quad y_1 = b \tan \theta$$

in the equation of the tangent at (x_1, y_1).

The equation of the normal at P is

$$a \cos \theta \,.\, x + b \cot \theta \,.\, y = a \cos \theta (a \sec \theta) + b \cot \theta (b \tan \theta)$$
$$= a^2 + b^2,$$

from which it can be shewn that four normals can be drawn to a hyperbola from a point in its plane.

183. The Asymptotes.

The asymptotes of a hyperbola are the pair of tangents drawn from its centre.

The equation of the pair of tangents from the centre of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is at once obtained by writing $x_1 = 0$, $y_1 = 0$, in the equation of the pair of tangents from (x_1, y_1), viz.,

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2.$$

We thus find that the asymptotes are given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

We must now set forth some particulars concerning them.

184. The equations of the asymptotes separately considered are

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0.$$

If now we take either of these equations and eliminate y between it and the equation of the curve we get $1 = 0$, which can never hold.

Thus it appears that the asymptotes do not meet the curve at all, and yet from § 181 it would seem that each of these lines should meet the curve in two coincident points.

We will endeavour now to give some explanation of this apparent inconsistency.

If we consider the line

$$y = \left(\frac{b}{a} - \epsilon\right)x,$$

where ϵ is very small, so that this line is inclined at a very small angle to the asymptote $y = \frac{b}{a}x$, we find that the x-co-

ordinates of the intersection of the line with the curve are given by the quadratic in x,

$$x^2\left\{\frac{1}{a^2}-\left(\frac{1}{a}-\frac{\epsilon}{b}\right)^2\right\}=1,$$

that is

$$x^2=\frac{1}{\frac{2\epsilon}{ab}-\frac{\epsilon^2}{b^2}}.$$

Now as ϵ is very small the values of x furnished by this equation are very large, so that we can say that the line

$$y=\left(\frac{b}{a}-\epsilon\right)x$$

meets the curve in two points equidistant from the origin and at a very great distance from it. The smaller ϵ is, the greater does this distance become.

The same thing is true of the line

$$y=-\left(\frac{b}{a}-\epsilon\right)x.$$

These two lines then tend as ϵ becomes smaller and smaller to become tangents to the hyperbola, for the points in which either of them meets the hyperbola are at a very great distance and they are on opposite sides of the centre. And it is one of the paradoxes of geometry with which the student is probably already acquainted that the points on a line in two opposite directions and at a very great distance tend to become the same point.

This is sometimes expressed by saying that the point at infinity in the one direction is the same as the point at infinity in the other.

Hence the lines

$$y=\left(\frac{b}{a}-\epsilon\right)x \text{ and } y=-\left(\frac{b}{a}-\epsilon\right)x,$$

when ϵ is infinitely small, are tangents to the hyperbola.

The lines then that we found to be the tangents from the centre in § 183, viz.,

$$y=\frac{b}{a}x \text{ and } y=-\frac{b}{a}x$$

are seen to be the 'limiting case' of these two lines

$$y=\pm\left(\frac{b}{a}-\epsilon\right)x$$

when ϵ is infinitely diminished.

The tangents from the centre of a hyperbola have their points of contact at an infinite distance, and it is to tangents with their points of contact 'at infinity' that the name 'asymptotes' is applied.

The lines called the asymptotes do not as we have seen meet the hyperbola at all, but the further on we go in imagination along them the nearer does the hyperbola approach them, and indeed the hyperbola tends to become these lines. We say 'tends to become,' for it never actually does so, and it is only loose speaking to say, as is sometimes done, that a curve becomes its asymptotes at infinity.

185. Asymptotes of an ellipse.

The equation of the pair of tangents from the centre of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is found exactly as in § 183, viz.,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0.$$

These lines are imaginary having no real point upon them except the origin. They possess however the same algebraical properties as the asymptotes of the hyperbola, and they are called the asymptotes of the ellipse. It is only because they are imaginary and not real that they have not the same geometrical importance.

186. *To find the equation of the asymptotes of a central conic given by the general equation,*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

We have seen in §§ 183, 185 that the equation of the asymptotes only differs from that of the curve in the constant term. Thus the equation of the asymptotes will be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \gamma = 0,$$

where γ is so chosen that *either* (1) this represents two straight lines, *or* (2) it passes through the centre of the conic. Both of these will give the same result.

Expressing the condition for two straight lines, we find that γ is given by

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c-\gamma \end{vmatrix} = 0,$$

that is
$$\begin{vmatrix} a, & h, & g \\ b, & h, & f \\ g, & f, & c \end{vmatrix} - \gamma \begin{vmatrix} a, & h, & 0 \\ h, & b, & 0 \\ g, & f, & 1 \end{vmatrix} = 0,$$

$$\therefore \gamma = \frac{\Delta}{ab - h^2},$$

where Δ stands for the determinant,

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

Thus the asymptotes are

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \frac{\Delta}{ab - h^2}.$$

Cor. The asymptotes of

$$(lx + my + n)(l'x + m'y + n') = k$$

are
$$(lx + my + n)(l'x + m'y + n') = 0.$$

187. Had we in the preceding article expressed the condition that (1) should go through the centre of the conic we should have obtained

$$\gamma = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c,$$

where (x_1, y_1) the centre is given by

$$ax_1 + hy_1 + g = 0 \quad \dots\dots\dots\dots\dots\dots\dots (1),$$

$$hx_1 + by_1 + f = 0 \quad \dots\dots\dots\dots\dots\dots\dots (2).$$

Now

$$\gamma = x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c),$$

$$\therefore \; gx_1 + fy_1 + c - \gamma = 0 \quad \dots\dots\dots\dots\dots\dots (3).$$

Eliminating (x_1, y_1) from (1), (2), (3) we get

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c-\gamma \end{vmatrix} = 0,$$

which is the same equation for γ as in the last article.

188. We might have obtained the asymptotes of the conic given by the general equation by writing the equation of the pairs of tangents from the centre (x_1, y_1). This is

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)$$
$$\times (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)$$
$$= \{(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + (gx_1 + fy_1 + c)\}^2,$$

where

$$ax_1 + hy_1 + g = 0 \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

$$hx_1 + by_1 + f = 0 \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

If then we write

$$gx_1 + fy_1 + c = \lambda \dots\dots\dots\dots\dots\dots\dots\dots (3),$$

$(1) \times x_1 + (2) \times y_1 + (3)$ gives

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = \lambda,$$

and the equation of the asymptotes is then

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \lambda,$$

where λ is found by eliminating (x_1, y_1) from (1), (2) and (3). That is

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c-\lambda \end{vmatrix} = 0,$$

which gives the same value for λ that we found for γ in § 186.

We have given these alternative methods that the student may see the matter from different points of view.

189. The equation of a pair of lines through the origin parallel to the asymptotes of the conic given by the general equation is (§ 58),

$$ax^2 + 2hxy + by^2 = 0,$$

and these lines are real if $h^2 > ab$, that is if the conic be a hyperbola, and they are imaginary if $h^2 < ab$, that is if the conic be an ellipse.

It will be found that this is a very simple way of remembering the criteria of discrimination of the ellipse and hyperbola. The conic is an ellipse or a hyperbola according as the lines

$$ax^2 + 2hxy + by^2 = 0$$

are imaginary or real.

190. The conjugate hyperbola.

If we have a hyperbola then the hyperbola, having for its transverse and conjugate axes the conjugate and transverse axes of the original hyperbola, is called the 'conjugate hyperbola.'

The equation of the hyperbola conjugate to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

is then

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

Both of these hyperbolas have the same asymptotes, viz.,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

191. *To find the equation of the hyperbola conjugate with the hyperbola given by the general equation*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

We have seen in § 190 that when the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

its conjugate is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0,$$

and its asymptotes are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Thus as the terms $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ common to these equations will for any change of origin and axes change into terms which will be the same for all three, if

$$H = 0, \qquad A = 0, \qquad H' = 0,$$

be the equations of a hyperbola, its asymptotes and its conjugate, then

$$H' - A = A - H.$$

That is, the equation of the conjugate will be

$$2A - H = 0.$$

Now in our case

$$H \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$

and $$A \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2}.$$

$\therefore$ The equation of the conjugate hyperbola is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{2\Delta}{ab - h^2} = 0.$$

Examples. 1. Find the asymptotes of the hyperbola

$$6x^2 - 7xy - 3y^2 + x + 4y = 0.$$

2. Find the asymptotes and the conjugate hyperbola of

$$2xy + 7x - 6y - 18 = 0.$$

3. Find the equation of the hyperbola conjugate with

$$(lx + my + n)(l'x + m'y + n') = k^2.$$

192. Conjugate diameters.

If through the centre C of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a line be drawn parallel to the tangent at P $(a \sec\theta, b \tan\theta)$, this line will meet the hyperbola in two imaginary points

$$(\pm ia\tan\theta, \quad \pm ib\sec\theta)$$

and the conjugate hyperbola in two real points

$$(\pm a\tan\theta, \quad \pm b\sec\theta).$$

The equation of the tangent at P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation of the line through the centre parallel to this is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Find where this meets the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

We have $$\frac{y}{b} = \frac{x}{a\sin\theta},$$

$$\therefore \quad \frac{x^2}{a^2}\left(1 - \operatorname{cosec}^2\theta\right) = 1.$$

$$\therefore \quad \frac{x^2}{a^2}\cot^2\theta = -1,$$

$$\therefore \quad x^2 = -a^2\tan^2\theta,$$

$$\therefore \quad x = \pm ia\tan\theta,$$

$$\therefore \quad y = \pm ib\sec\theta.$$

Now find where (2) meets the conjugate hyperbola

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

We get at once $$x^2 = a^2 \tan^2 \theta,$$

$$\therefore \; x = \pm a \tan \theta,$$

and $$y = \pm b \sec \theta.$$

Thus the proposition is proved.

193. Let D_1, D_1' be the points in which (2) meets the original hyperbola, and let D and D' be the points in which it meets the conjugate hyperbola,

$$\therefore \; CD^2 = - CD_1^2.$$

Now the equation of the tangent at D_1 $(ia \tan \theta, \; ib \sec \theta)$ to the original hyperbola is

$$\frac{x \, . \, i \tan \theta}{a} - \frac{y \, . \, i \sec \theta}{b} = 1.$$

The equation of a line through the centre parallel to this is

$$\frac{x \tan \theta}{a} - \frac{y \sec \theta}{b} = 0,$$

and this is satisfied by coordinates of P $(a \sec \theta, \, b \tan \theta)$,
$\therefore$ CP is parallel to the tangents at D_1, D_1' to the original hyperbola.

Again the tangent at D $(a \tan \theta, \, b \sec \theta)$ to the conjugate hyperbola is

$$\frac{y \sec \theta}{b} - \frac{x \tan \theta}{a} = 1.$$

The line through the centre parallel to this is

$$\frac{y \sec \theta}{b} - \frac{x \tan \theta}{a} = 0,$$

which is satisfied by the point P.

Hence the tangents at D and D' on the conjugate hyperbola are parallel to CP.

We might then speak of CP, CD_1 as conjugate semidiameters or of CP, CD as conjugate semidiameters.

Because D is real it is more usual to take CP, CD as the conjugate semidiameters. But it must be remembered when this is done that the extremities of one of two conjugate diameters lie on the original hyperbola and those of the other on the conjugate hyperbola.

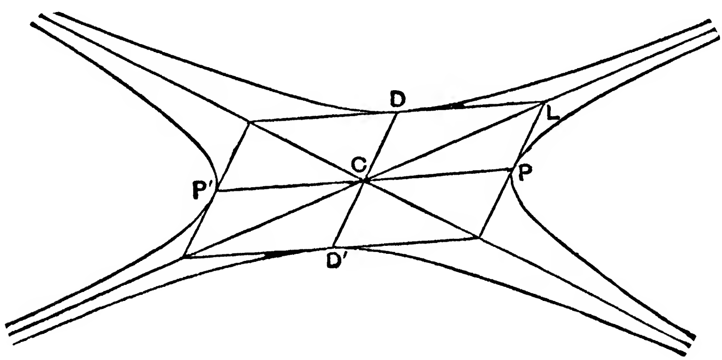
If PCP', DCD' be conjugate diameters,

$$CP^2 - CD^2 = (a^2 \sec^2\theta + b^2 \tan^2\theta) - (a^2 \tan^2\theta + b^2 \sec^2\theta)$$
$$= a^2 - b^2.$$

Whereas $$CP^2 + CD_1^2 = a^2 - b^2.$$

This last result we could have surmised from the fact that in the ellipse the sum of the squares of two conjugate diameters $= a^2 + b^2$. The corresponding property of the hyperbola is obtained by writing $-b^2$ for b^2.

194. *The vertices of the parallelogram formed by drawing tangents at the extremities of two conjugate diameters lie on the asymptotes, and the area of this parallelogram is constant* $= 4ab$.



Let PCP', DCD' be the conjugate diameters.

Let P be $(a \sec\theta, b \tan\theta)$, P' $(-a \sec\theta, -b \tan\theta)$.

Then D is $(a \tan\theta, b \sec\theta)$, D' $(-a \tan\theta, -b \sec\theta)$.

The tangent at P is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

The tangent at D is

$$-\frac{x}{a}\tan\theta + \frac{y}{b}\sec\theta = 1.$$

Where these meet

$$\left(\frac{x}{a} - \frac{y}{b}\right)(\sec\theta + \tan\theta) = 0,$$

$$\therefore \frac{x}{a} - \frac{y}{b} = 0.$$

Thus the intersection of tangents at P and D lies on one asymptote.

Similarly the intersections of tangents at P' and D' lie on the same asymptote, and the intersections of tangents at P' and D, and of tangents at P and D' lie on the other asymptote.

Let tangents at P and D meet in L.

Then area of parallelogram formed by the tangents at

$$P, P', D, D' = 4 \text{ area } DCPL$$

$$= 4CD \times \text{perpendicular from } C \text{ on } PL$$

$$= 4\sqrt{a^2\tan^2\theta + b^2\sec^2\theta} \times \frac{1}{\sqrt{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}}$$

$$= 4ab,$$

which is constant.

Cor. *The portion of a tangent to a hyperbola intercepted between the asymptotes is bisected at the point of contact, and the area of the triangle formed by this tangent and the two asymptotes is constant* $(= ab)$.

195. The student will see that § 174 is applicable to the hyperbola by merely writing $-b^2$ for b^2.

The conditions that $y = mx$, $y = m'x$ should be conjugate diameters of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } mm' = \frac{b^2}{a^2}.$$

Also each of two conjugate diameters bisects all chords parallel to the other.

We can at once obtain the equation of the hyperbola referred to a pair of conjugate diameters of lengths $2a'$, $2b'$ as

$$\frac{x'^2}{a'^2} - \frac{y'^2}{b'^2} = 1.$$

For the equation is of the form

$$Ax'^2 + 2Hx'y' + By'^2 = 1.$$

But to every point (x', y') there corresponds a point $(x', -y')$.

$$\therefore H = 0,$$

and when $y' = 0,\ x'^2 = a'^2,$

and when $x' = 0,\ y'^2 = -b'^2,$

whence we get

$$A = \frac{1}{a'^2}, \quad B = -\frac{1}{b'^2}.$$

196. Equation of hyperbola referred to its asymptotes as axes.

A hyperbola which has the axes of coordinates for asymptotes has its equation of the form

$$xy = \text{a constant}. \qquad (\S\ 186\ \text{Cor.})$$

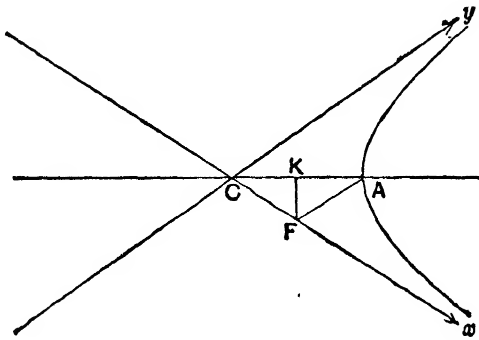
To find the value of the constant when the axes are of lengths $2a$, $2b$ we consider the coordinates of the vertex A of the hyperbola.

Draw AF parallel to the y-axis to meet the x-axis in F, and draw FK perpendicular to CA,

$$\therefore \quad CK = \frac{a}{2},$$

and $$x_A = CF = CK \sec\alpha = \frac{a}{2}\sec\alpha,$$

where α is the angle the asymptotes make with the transverse axis so that $\tan\alpha = \frac{b}{a}$.



Thus

$$x_A y_A = x_A^2 = \frac{a^2}{4}\sec^2\alpha = \frac{a^2}{4}\left(1 + \frac{b^2}{a^2}\right) = \frac{a^2 + b^2}{4}.$$

Hence the equation of the hyperbola referred to its asymptotes is

$$xy = \frac{a^2 + b^2}{4}.$$

The equation of the conjugate hyperbola is easily seen to be

$$xy = -\frac{a^2 + b^2}{4}.$$

197. The equation of a hyperbola referred to its asymptotes being $xy = k^2$ the tangent at (x_1, y_1) is

$$\tfrac{1}{2}(xy_1 + x_1y) = k^2,$$

that is $$\frac{x}{2x_1} + \frac{y}{2y_1} = \frac{k^2}{x_1y_1} = 1.$$

Thus the intercepts on the axes are $2x_1$, $2y_1$.

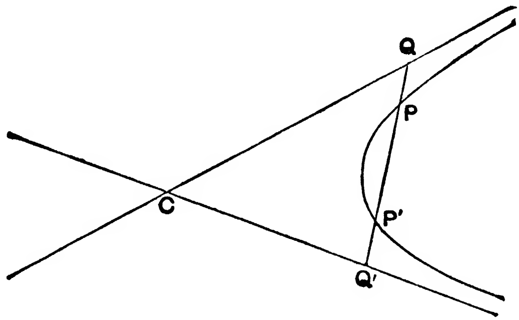
This proves again that the part of the tangent intercepted between the asymptotes is bisected at the point of contact.

198. Again, the equation of the chord of $xy = k^2$ whose middle point is (x_1, y_1) is (§ 141)

$$xy_1 + x_1y = 2x_1y_1,$$

that is

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1.$$



The intercepts on the axes are then $2x_1$, $2y_1$.

This shews that the middle point of the chord is also the middle part of that portion of the chord which is intercepted between the asymptotes.

Thus if the chord PP' meets the asymptotes in Q and Q',

$$QP = P'Q'.$$

This is a well known property of the hyperbola.

EXAMPLES.

1. Hyperbolas are drawn through the origin O with one focus at a fixed point (h, k) and one asymptote parallel to OX, prove that the locus of their centres is

$$\{h(x-h) + y(y-k)\}^2 = (x-h)^2(h^2+k^2).$$

2. The four normals are drawn from any point to the rectangular hyperbola $xy = c^2$. If the tangents at the feet of two of the normals meet in (ξ_r, η_r); $r = 1, 2 \ldots 6$, shew that

$$\sum_{r=1}^{r=6} (\xi_r/\eta_r) = \sum_{r=1}^{r=6} (\eta_r/\xi_r) = 0$$

and

$$\xi_1\xi_2\ldots\xi_6 + \eta_1\eta_2\ldots\eta_6 = 0.$$

3. Shew that the distance between the points of contact of a common tangent to two rectangular hyperbolas the axes of one of which coincide with the asymptotes of the other is

$$\frac{1}{aa'}(a^4 + a'^4)^{\frac{3}{4}}$$

where $2a$ and $2a'$ are the transverse axes.

4. Any tangent to the hyperbola $4xy = ab$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points P and Q; shew that the normals to the ellipse at P and Q meet on a fixed diameter of the ellipse.

5. Chords of a rectangular hyperbola at right angles to one another subtend a right angle at a fixed point O, shew that they intersect on the polar of O.

6. Shew how to find the coordinates of the vertices of a triangle inscribed in the hyperbola $xy = k^2$, the sides of the triangle being parallel to the lines $y + lx = 0$, $y + mx = 0$, $y + nx = 0$. Shew that if l, m, n vary the area of the triangle is always proportional to

$$(m - n)(n - l)(l - m) \div lmn.$$

7. Interpret the equation

$$(Ax + By + C)(Bx - Ay + D) = A^2 + B^2.$$

8. Shew that the locus of the intersection of tangents to a hyperbola inclined to one another at the same angle as the asymptotes is the inverse of an ellipse with regard to its centre.

9. At the point of intersection of the rectangular hyperbola $xy = k^2$ and of the parabola $y^2 = 4ax$, the tangents to the hyperbola and parabola make angles θ and ϕ respectively with the axis of x. Prove

$$\tan\theta = -2\tan\phi.$$

10. A rectangular hyperbola is cut by any circle in four points. Prove that the sum of the squares of the distances of these four points from the centre of the hyperbola is equal to the square on the diameter of the circle.

11. Through any point on the polar of P with respect to a rectangular hyperbola, two chords are drawn each subtending a right angle at P. Prove that the chords are at right angles.

12. A normal to a hyperbola meets the conjugate axis in P and the transverse axis in Q. Shew that if tangents be drawn from P to the hyperbola meeting the circle described on PQ as diameter in T and T', TT' will touch the hyperbola.

13. Shew that the circles whose diameters are chords of a rectangular hyperbola drawn parallel to a given direction constitute a coaxial system; and that the systems corresponding to two directions at right angles are orthogonal to one another.

14. The four normals at the points of a rectangular hyperbola $xy=c^2$ in which it is met by the chords

$$x\cos\alpha + y\sin\alpha - p = 0, \quad p\,(x\sin\alpha - y\cos\alpha) - c^2\cos 2\alpha = 0$$

are concurrent.

15. Shew that the part of a common tangent of the conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{a^2+b^2}{a^2-b^2}=0$$

intercepted between the points of contact subtends a right angle at the centre.

16. A rectangular hyperbola passes through two fixed points and its asymptotes are in given directions. Prove that its vertices lie on an ellipse and on a hyperbola which intersect orthogonally.

17. Prove that the polar of any point on an asymptote of a hyperbola with respect to the hyperbola is parallel to that asymptote.

18. Prove that any tangent to either of the conics

$$\frac{x^2}{a^3}+\frac{y^2}{b^3}=\frac{1}{a+b} \text{ and } \frac{x^2}{a^3}-\frac{y^2}{b^3}=\frac{1}{a-b},$$

meets the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ in two points the normals at which are equidistant from the centre.

19. The normal at P_1 to the rectangular hyperbola $xy=c^2$ meets the curve again at P_2, the normal at P_2 meets the curve again in P_3 and so on. Prove that if $y_1 y_2 y_3 \ldots$ are the ordinates of these points

$$y_1^3 y_2 = y_2^3 y_3 = \ldots = -c^4.$$

CHAPTER XI.

POLAR EQUATIONS OF CONICS.

199. The polar equation of a conic can always be found from the corresponding Cartesian equation referred to rectangular coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$. Thus, for example, the polar equation of an ellipse of axes $2a$, $2b$, the centre being the pole and the major axis the initial line, is

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

this equation being obtained by the above substitution from the standard equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If, however, the major axis make an angle α with the initial line the polar equation will be

$$\frac{r^2\cos^2(\theta-\alpha)}{a^2} + \frac{r^2\sin^2(\theta-\alpha)}{b^2} = 1.$$

In the same way from the equation $y^2 = 4ax$ of the parabola we can obtain the corresponding polar equation

$$r\sin^2\theta = 4a\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

As the line $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$, it follows that the line

$$\frac{a}{r} = m\sin\theta - m^2\cos\theta$$

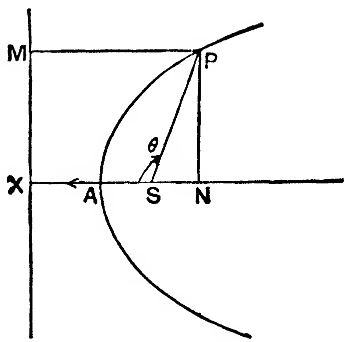
is a tangent to (2) whatever constant value m may have.

As a matter of fact the polar equations of conics are not much used except in the case where the pole is at a focus of the conic. In this case the polar equation assumes a very simple form which is frequently of use. This equation we shall now proceed to find independently of any Cartesian equation.

200. Polar Equation with focus as pole.

To find the polar equation of a conic when the focus S is taken for pole, and the line SA joining S to the corresponding vertex A is the initial line.

Let P be any point on the conic. Draw PM perpendicular to the directrix and PN perpendicular to SA. Let r and θ be the polar coordinates of P. Let $l =$ semi-latus rectum.



Then $$r = SP = e\,.\,PM$$
$$= e\,.\,XN$$
$$= e\,(XS + SN)$$
$$= l + e\,.\,r\cos(\pi - \theta) = l - er\cos\theta;$$
$$\therefore\ r(1 + e\cos\theta) = l,$$

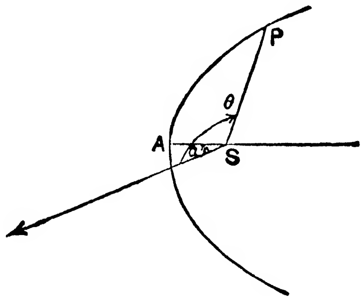
that is $$\frac{l}{r} = 1 + e\cos\theta.$$

If instead of SA for initial line we take the line opposite to SA the equation becomes $\dfrac{l}{r} = 1 - e\cos\theta.$

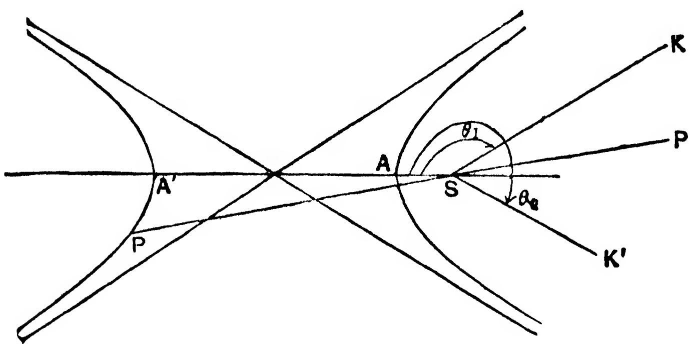
If however the line SA make an angle α with the initial line the equation of the conic takes the form

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha),$$

for the angle ASP is now $\theta - \alpha$.



201. It must be observed that if the conic be a hyperbola the equation $\frac{l}{r} = 1 + e\cos\theta$ is only satisfied by points on the *further* branch of the curve if the radius vector is negative. That is if P be a point on the further branch the vectorial angle of P is not ASP but ASP' where P' is on PS produced.



The vectorial angle of the points at infinity on the curve are given by $\cos\theta = -\frac{1}{e}$, in other words the two lines drawn through the focus and whose vectorial angles satisfy this relation

are parallel to the asymptotes. Denoting these angles by θ_1 and θ_2 we see that when θ lies between θ_1 and θ_2, r is negative if the equation is to be satisfied.

The student will do well to revert to what was said in § 6 on the sign of the radius vector. By the admission of negative radii vectores the equation $\frac{l}{r} = 1 + e\cos\theta$ represents both branches of the hyperbola.

202. Equation of Chord.

To find the equation of the line cutting the conic

$$\frac{l}{r} = 1 + e\cos\theta$$

in points whose vectorial angles are $\alpha - \beta$ *and* $\alpha + \beta$.

The general equation of a line not through the pole can be written

$$\frac{l}{r} = A\cos\theta + B\sin\theta.$$

If this passes through the points on the conic whose vectorial angles are $\alpha - \beta$, $\alpha + \beta$,

$$1 + e\cos(\alpha - \beta) = A\cos(\alpha - \beta) + B\sin(\alpha - \beta),$$

$$1 + e\cos(\alpha + \beta) = A\cos(\alpha + \beta) + B\sin(\alpha + \beta).$$

These equations determine A and B.

We can write them

$$\left.\begin{array}{l}(A - e)\cos(\alpha - \beta) + B\sin(\alpha - \beta) = 1\\(A - e)\cos(\alpha + \beta) + B\sin(\alpha + \beta) = 1\end{array}\right\}.$$

Whence we have

$$(A - e)\sin(\alpha + \beta - \overline{\alpha - \beta}) = \sin(\alpha + \beta) - \sin(\alpha - \beta),$$

$$\therefore\ 2(A - e)\sin\beta\cos\beta = 2\cos\alpha\sin\beta,$$

$$\therefore\ A = e + \cos\alpha\sec\beta$$

and
$$B\sin 2\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta),$$

$$\therefore\ B = \sin\alpha\sec\beta.$$

Hence the equation required is

$$\frac{l}{r} = (e + \cos\alpha \sec\beta)\cos\theta + \sin\alpha \sec\beta \sin\theta,$$

that is
$$\frac{l}{r} = e\cos\theta + \sec\beta \cos(\theta - \alpha).$$

203. Tangent and Normal.

The equation of the tangent at the point whose vectorial angle is α is at once obtained by putting $\beta = 0$, viz.

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha),$$

for this line meets the conic in two coincident points whose vectorial angle is α.

The equation of the normal will be of the form

$$\frac{kl}{r} = e\cos\left(\frac{\pi}{2} + \theta\right) + \cos\left(\frac{\pi}{2} + \theta - \alpha\right) \quad (\S\ 48)$$

$$= -e\sin\theta - \sin(\theta - \alpha)$$

and k must be so chosen that this shall pass through the point given by

$$\theta = \alpha \text{ and } \frac{l}{r} = 1 + e\cos\alpha,$$

that is
$$k(1 + e\cos\alpha) = -e\sin\alpha.$$

Hence the equation of the normal is

$$\frac{e\sin\alpha}{1 + e\cos\alpha} \cdot \frac{l}{r} = e\sin\theta + \sin(\theta - \alpha).$$

204. Chord of Contact.

To find the equation of the chord of contact of tangents from the point (r_1, θ_1).

Let $\alpha - \beta$, $\alpha + \beta$ be the vectorial angles of the points of contact.

The equation of the chord is then

$$\frac{l}{r} = e\cos\theta + \sec\beta \cos(\theta - \alpha) \quad \text{.................(1).}$$

Now the tangent at $\alpha-\beta$ is

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\overline{\alpha-\beta}).$$

This passes through (r_1, θ_1),

$$\therefore\quad \frac{l}{r_1}=e\cos\theta_1+\cos(\theta_1-\overline{\alpha-\beta}).$$

Similarly $$\frac{l}{r_1}=e\cos\theta_1+\cos(\theta_1-\overline{\alpha+\beta}),$$

whence we get $$\cos(\theta_1-\overline{\alpha-\beta})=\cos(\theta_1-\overline{\alpha+\beta});$$

$$\therefore\quad \theta_1-(\alpha-\beta)=2n\pi\pm\{\theta_1-\overline{\alpha+\beta}\}.$$

The lower sign must be taken here, since the upper sign would give a special value of β,

$$\therefore\quad \theta_1-(\alpha-\beta)=2n\pi-\{\theta_1-(\alpha+\beta)\},$$

$$\therefore\quad \theta_1=n\pi+\alpha.$$

This gives α and we have

$$\begin{aligned}\frac{l}{r_1}-e\cos\theta_1&=\cos(n\pi+\beta)\\&=(-1)^n\cos\beta,\end{aligned}$$

$\therefore$ (1) becomes

$$\begin{aligned}\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)&=(-1)^n\cos(\theta-\overline{\theta_1+n\pi})\\&=\cos(\theta-\theta_1).\end{aligned}$$

205. From the last article we can see that *tangents from a point to a conic subtend at a focus angles which are equal or supplementary.*

For let TP and TQ be tangents from T. Let $\alpha-\beta$, $\alpha+\beta$ be the vectorial angles of P and Q, and let θ_1 be that of T.

By § 204 $$\theta_1-\alpha=n\pi.$$

Thus $$\theta_1-(\alpha-\beta)=n\pi+\beta$$

and $$(\alpha+\beta)-\theta_1=\beta-n\pi,$$

and these differ by a multiple of 2π for

$$n\pi+\beta-(\beta-n\pi)=2n\pi.$$

Thus the angle $PST =$ angle TSQ, unless the curve be a hyperbola and the tangents be drawn to different branches, in which case if TQ be the tangent to the further branch and QS be produced to Q', ASQ' is the vectorial angle of Q (§ 201).

So that $$\angle PST = \angle TSQ'.$$

Therefore the angles subtended by TP and TQ at S are supplementary.

If *both* the tangents TP, TQ touch the nearer or *both* the further branch TP and TQ subtend equal angles at S.

206. **Prop.** *The semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord.*

Let PSQ be a focal chord.

Let θ be the vectorial angle of P.

$\therefore$ $\pi + \theta$ is the vectorial angle of Q,

$$\therefore \frac{l}{SP} = 1 + e\cos\theta, \quad \frac{l}{SQ} = 1 + e\cos(\pi + \theta) = 1 - e\cos\theta,$$

that is $$\frac{l}{SP} + \frac{l}{SQ} = 2;$$

$$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l},$$

that is l, the semi-latus rectum, is a harmonic mean between SP and SQ.

Should it happen that the curve is a hyperbola and that P is on the nearer branch while Q is on the further one, then if θ is the vectorial angle of P, $\pi + \theta$ is still the vectorial angle of Q (§ 201), but the radius vector of Q is the negative of the numerical value of SQ, so that we now have

$$\frac{l}{SP} = 1 + e\cos\theta, \quad -\frac{l}{SQ} = 1 - e\cos\theta,$$

$$\therefore \frac{1}{SP} - \frac{1}{SQ} = \frac{2}{l}.$$

A more comprehensive statement of the proposition then would be:

The semi-latus rectum of any conic is a harmonic mean between the algebraical focal distances of the extremities of a focal chord, it being understood that a focal distance is negative if the point be on the further branch, otherwise it is positive.

EXAMPLES.

1. PQ is a variable chord of a conic having a focus at S and the angle PSQ is constant; prove that the locus of the intersection of the tangents at P and Q is a conic having S for a focus, and the corresponding directrix in common with the given conic.

2. The general equation of (i) the chord, (ii) the tangent, (iii) the normal of the circle $r = 2a\cos\theta$ are given by

$$\text{(i)}\quad r\cos(\alpha+\beta-\theta) = 2a\cos\alpha\cos\beta,$$
$$\text{(ii)}\quad r\cos(2\alpha-\theta) = 2a\cos^2\alpha,$$
$$\text{(iii)}\quad r\sin(2\alpha-\theta) = 2a\sin\alpha\cos\alpha$$

respectively.

3. A series of rectangular hyperbolas have a given focus and pass through a given point; prove that the locus of the other focus when referred to the given focus as origin can be expressed by an equation of the form $r = a\cos\theta + b$ where a and b are constant.

4. The eccentric angle of any point P on an ellipse is α, measured from the semi-major axis CA, S is the focus nearest to A and the angle $ASP = \theta$; prove

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}.$$

[This relation is of importance in the theory of elliptic orbits in dynamics.]

5. P, Q, R are three points on the conic $\frac{l}{r} = 1 + e\cos\theta$, the focus S being the pole; SP and SR meet the tangent at Q in M and N so that $SM = SN = l$. Prove that PR touches the conic

$$\frac{l}{r} = 1 + 2e\cos\theta.$$

6. Two parabolas have a common focus and axes inclined at an angle α. Prove that the locus of the intersection of two perpendicular tangents one to each of the parabolas is a conic.

7. Chords of a conic which subtend a constant angle at a focus touch a fixed conic.

8. The equation of the circle circumscribing the triangle formed by the tangents at the three points on the parabola $\frac{l}{r} = 1 + \cos\theta$ whose vectorial angles are α, β, γ is

$$2r \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2} = l \cos\left(\theta - \frac{\alpha+\beta+\gamma}{2}\right).$$

9. Shew that the sum of the reciprocals of the areas of the rectangles formed by the segments of any two perpendicular focal chords of a conic is constant.

10. The equation of the director circle of the conic

$$\frac{l}{r} = 1 + e\cos\theta$$

is $$r^2(1-e^2) + 2elr\cos\theta - 2l^2 = 0.$$

11. A circle is drawn through a focus of a conic whose latus rectum is $2l$; shew that the sum of the reciprocals of the focal distances of the four points in which the circle cuts the conic is $\frac{2}{l}$.

12. Find the condition that the line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$.

13. Conics with latus rectum of given length are described with a fixed point as focus and touching a fixed straight line. Prove that the locus of their centres is a conic.

14. Points P and Q are taken on a conic in such a manner that the vectorial angle of the point of intersection of the circles on the focal radii SP, SQ as diameters has a constant value κ. Shew that the locus of the pole of PQ is the line

$$\frac{l}{r} = e\cos\theta + \frac{\sin(\theta-\kappa)}{e\sin\kappa}.$$

15. The equation of the tangent at (r_1, θ_1) to a circle whose centre is (c, α) is

$$r_1^2 = r_1 c \cos(\theta_1 - \alpha) - cr\cos(\alpha - \theta) + rr_1\cos(\theta - \theta_1) = 0.$$

16. If θ, θ' are the vectorial angles of any point on a given conic referred to the two foci, the initial line in both cases being the axis in the same sense, then the ratio $\tan\frac{\theta}{2} : \tan\frac{\theta'}{2}$ is constant.

17. Prove that the equations of the asymptotes of the hyperbola $\frac{l}{r} = 1 + e\cos\theta$ are

$$\frac{l}{r} = \frac{e^2 - 1}{e}\left\{\cos\theta \pm \frac{\sin\theta}{\sqrt{e^2 - 1}}\right\}.$$

Find too the asymptotes of $\frac{l}{r} = 1 + e\cos(\theta - \gamma)$.

18. If the focus of a conic be given and if the asymptotes pass each through a fixed point on a straight line through the focus, the locus of the centre will be a circle.

19. An ellipse and a parabola have a common focus S and intersect in two real points P and Q, of which P is the vertex of the parabola. If e be the eccentricity of the ellipse and α the angle which SP makes with the major axis, prove that

$$\frac{SQ}{SP} = 1 + \frac{4e^2\sin^2\alpha}{(1 - e\cos\alpha)^2}.$$

20. A family of conics have a common focus, axes in the same direction and equal latera recta. Shew that the locus of the foot of the perpendicular from the focus on a common tangent to any pair whose eccentricities are connected by the homographic relation

$$aee' + b(e + e') + c = 0$$

is a circle.

CHAPTER XII.

CONICS IN GENERAL.

207. Proposition. *Two real conics will in general cut in four and only four points, an even number of which may be imaginary.*

For we may take as the equations of our conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{............(1)},$$

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \quad \text{............(2)}.$$

To find where these conics meet, we treat the equations as simultaneous.

We may write the equations

$$A_0y^2 + A_1y + A_2 = 0,$$

$$B_0y^2 + B_1y + B_2 = 0,$$

where the coefficients are functions of x of degrees indicated by their suffixes.

From these we have

$$\frac{y^2}{\begin{vmatrix} A_1, & A_2 \\ B_1, & B_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} A_0, & A_2 \\ B_0, & B_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} A_0, & A_1 \\ B_0, & B_1 \end{vmatrix}},$$

whence

$$\begin{vmatrix} A_1, & A_2 \\ B_1, & B_2 \end{vmatrix} \begin{vmatrix} A_0, & A_1 \\ B_0, & B_1 \end{vmatrix} = \begin{vmatrix} A_0, & A_2 \\ B_0, & B_2 \end{vmatrix}^2,$$

which is a biquadratic in x, having four roots, and to each value of x corresponds *one* value of y given by

$$y = -\begin{vmatrix} A_0, & A_2 \\ B_0, & B_2 \end{vmatrix} \div \begin{vmatrix} A_0, & A_1 \\ B_0, & B_1 \end{vmatrix}.$$

Thus the conics will in general cut in four points some of which may be coincident. Two of the points or all four of them may be imaginary, for imaginary roots of an algebraical equation with real coefficients occur in pairs.

208. Should it happen that each of the conics (1) and (2) is a pair of straight lines, and that one line of each pair is the same, then clearly there would be an infinite number of points common to the two conics, namely all points on the line common to them.

For the equations (1) and (2) now take the form

$$(lx + my + n)(l'x + m'y + n') = 0,$$
$$(lx + my + n)(l''x + m''y + n'') = 0,$$

and these are satisfied by all points satisfying

$$lx + my + n = 0,$$

as well as by the point determined by

$$\left.\begin{aligned} l'x + m'y + n' &= 0 \\ l''x + m''y + n'' &= 0 \end{aligned}\right\}.$$

Again in the special case where (1) and (2) both represent two straight lines, and the point of intersection of the first pair is the same as that of the second pair, namely (x_1, y_1), the equations can be written in the form

$$\{l\ (x - x_1) + m\ (y - y_1)\}\{l'\ (x - x_1) + m'\ (y - y_1)\} = 0,$$
$$\{l''(x - x_1) + m''(y - y_1)\}\{l'''(x - x_1) + m'''(y - y_1)\} = 0,$$

which are satisfied simultaneously only by $x = x_1$, $y = y_1$.

209. Contact of Conics.

We will denote the four common points of two conics S and S' by Q, R, T, U. Now it may happen that two or more of these points coincide. Suppose that Q and R coincide

while T and U are separate points. The conics are then said to touch at the point Q, or to have 'single contact.'

Suppose now that Q and R coincide, as also T and U, but Q and T do not coincide. The conics then touch at two points Q and T and are said therefore to have 'double contact.'

Suppose next that Q, R and T coincide but U is a separate point. The conics are then said to have 'three point contact' at Q.

'Three point contact' is sometimes called 'contact of the second order' but it must be most carefully discriminated from 'double contact.'

Lastly suppose that Q, R, T and U all coincide, the conics are then said to have 'four point contact,' or as it is sometimes called 'contact of the third order.'

When two conics have contact of any order at a point they will have a common tangent line at that point.

Conics which have single contact may be looked upon as the limiting case of two conics which cut in four points, two of which are very near together. Such conics are sometimes said to have two 'consecutive points' common. So conics having three point contact may be regarded as the limiting case of two conics which cut in four points, three of which are very near together. Such conics are sometimes said to have three 'consecutive points' common.

And in the same way conics with four point contact may be said to have four 'consecutive points' common.

We shall return to the subject of the contact of conics later.

210. Proposition. *One conic, and in general only one, can be drawn through five given points.*

For, as the general equation of a conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

if we express the condition that this shall pass through the five points whose coordinates are supposed given, we shall have five

simple equations in a, b, c, f, g, h which will in general determine uniquely the ratios of these six constants to one another.

The five equations may however be not all independent. This case will clearly arise when as many as four of the five given points are collinear. For suppose we have four of the points in a line. Take the axis of x to be this line. Then we may take the points to be $(0, 0)$, $(x_1, 0)$, $(x_2, 0)$, $(x_3, 0)$.

The conic will now be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0,$$

and we must have

$$ax_1^2 + 2gx_1 = 0 \quad\text{.......................}(1),$$

$$ax_2^2 + 2gx_2 = 0 \quad\text{.......................}(2),$$

$$ax_3^2 + 2gx_3 = 0 \quad\text{.......................}(3).$$

Since x_1 is not equal to x_2 nor is either of these zero, (1) and (2) give $a = 0 = g$.

Thus the conic is

$$2hxy + by^2 + 2fy = 0,$$

and (3) is satisfied of itself.

We now express the condition that the remaining point, (x_4, y_4), should lie on the conic, and we have

$$2hx_4y_4 + by_4^2 + 2fy_4 = 0,$$

which is not sufficient to determine the ratio $b : f : h$.

Thus there will be an infinite number of conics through the five points, viz. the line through the four collinear points, together with *any* line through the remaining fifth point.

If only three of the points are collinear, there will be only one conic through the five points, viz. the line containing these three collinear points together with the line containing the other two.

Our proposition then is proved that one conic can always be found to pass through five points and in general only *one* such conic can be found, the exceptional case being when as many as four of the points are collinear.

211. Conics through the points common to two conics.

If
$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ ......(1)},$$
$$S' \equiv a'x^2 + h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \text{ ......(2)}$$
be two conics, then
$$S - kS' = 0, \text{ ....(3)}$$
where k is any constant, will be a conic, and it will pass through the points common to (1) and (2), since (3) is satisfied by $S = 0$ and $S' = 0$ simultaneously.

We see then that (3), for different values of k, gives all the conics passing through the four points common to (1) and (2). For any conic through these four points is (by § 210) completely determined when a fifth point on the conic is known. And the constant k can be so determined that the conic shall pass through this fifth point.

In the special case where (1) and (2) are both of them pairs of lines with a line in common, (3) will be a pair of lines one of which is this common line.

212. From the preceding paragraph we see that if $S = 0$ be a conic, then
$$S - k(lx + my + n)(l'x + m'y + n') = 0 \text{ ....(1)}$$
for any constant value of k will be a conic through the four points in which the lines
$$lx + my + n = 0,$$
$$l'x + m'y + n' = 0$$
meet the conic.

And in particular
$$S - k(lx + my + n)^2 = 0 \text{ ....(2)}$$
will be a conic touching the conic $S = 0$ at each of the points in which the line $lx + my + n = 0$ meets it. This is so for
$$(lx + my + n)^2 = 0$$
is a conic meeting $S = 0$ in four points, which are two pairs of coincident points.

Thus (2) will meet $S=0$ in two pairs of coincident points, that is to say (2) will have *double contact* with the conic $S=0$ at the two points where the line $lx+my+n=0$ meets it.

In the special case where the line $lx+my+n=0$ is a tangent to $S=0$, the conic (2) will have four point contact with $S=0$.

213. Proposition. *The common chords of a conic and circle taken in pairs are equally inclined to the axes of the conic.*

We shall take the axes of coordinates to be parallel to the axes of the conic so that the term in xy will disappear and the equation of the conic will be of the form

$$ax^2+by^2+2gx+2fy+c=0 \quad \text{..............(1)},$$

a or b being zero in the special case where the conic is a parabola.

Now let a circle cut the conic in four points P, Q, R, S and let the equations of a pair of the common chords, say PQ and RS, be

$$lx+my+n=0 \quad \text{....................(2)},$$

$$l'x+m'y+n'=0 \quad \text{....................(3)}.$$

Then

$$ax^2+by^2+2gx+2fy+c$$
$$-k(lx+my+n)(l'x+m'y+n')=0 \text{...(4)}$$

is the general equation of conics through the points of intersection of (2) and (3) with (1).

Therefore by properly choosing k, (4) will represent the circle. For this the coefficient of xy in (4) must be zero, that is

$$lm'+l'm=0,$$

$$\therefore \frac{l'}{m'}=-\frac{l}{m}.$$

Thus the 'm' of PQ is the negative of the 'm' of RS, that is the chords PQ and RS are equally inclined, but in opposite directions, to the x-axis.

That is, PQ, RS are equally inclined to the axes of the conic. Similarly the other pairs of chords, viz. PR, QS and PS, QR are equally inclined to the axes.

COR. *If α, β, γ, δ be the eccentric angles of the four points in which a circle cuts an ellipse*

$$\alpha+\beta+\gamma+\delta = \textit{an even multiple of } \pi.$$

For if we refer the ellipse to its principal axes the chords through α, β, and γ, δ are respectively

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}=0,$$

$$\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}=0.$$

Therefore, by properly choosing the constant k, the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+k\left(\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right)$$
$$\times\left(\frac{x}{a}\cos\frac{\gamma+\delta}{2}+\frac{y}{b}\sin\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}\right)=0$$

can be made to represent the circle through the four points α, β, γ, δ. For this the coefficient of xy must be zero,

$$\therefore\ \cos\frac{\alpha+\beta}{2}\sin\frac{\gamma+\delta}{2}+\cos\frac{\gamma+\delta}{2}\sin\frac{\alpha+\beta}{2}=0,$$

that is
$$\sin\frac{\alpha+\beta+\gamma+\delta}{2}=0,$$

$$\therefore\ \frac{\alpha+\beta+\gamma+\delta}{2}=\text{a multiple of } \pi,$$

$$\therefore\ \alpha+\beta+\gamma+\delta=\text{an even multiple of } \pi.$$

Examples. 1. Every conic through the four points in which a circle cuts an ellipse will have its axes parallel to the axes of the ellipse.

2. A circle cuts the parabola $y^2=4ax$ in the points $(a\mu_1^2, 2a\mu_1)$, $(a\mu_2^2, 2a\mu_2)$, $(a\mu_3^2, 2a\mu_3)$, $(a\mu_4^2, 2a\mu_4)$, prove that

$$\mu_1+\mu_2+\mu_3+\mu_4=0.$$

3. Three normals are drawn to a parabola from a point. Prove that the circle through the feet of these normals passes through the vertex of the parabola.

4. Two points, of eccentric angles γ' and δ', on an ellipse, are such that there exist a pair of points on the ellipse which are concyclic with them and are also concyclic with the pair of points α and β, which again are concyclic with γ and δ. Prove that γ, δ, γ', δ' are concyclic.

5. Prove that the conics

$$7y^2 - 4xy + 4x + 2y - 17 = 0,$$
$$x^2 + 2y^2 - 4 = 0,$$

touch each other at two distinct points, and find the coordinates of the intersection of the tangents at those points.

6. Shew that it is possible to describe a circle touching the two lines

$$(ax^2 + by^2)(l^2 - m^2) + 2(a - b)\,lmxy = 0,$$

where they are met by the line $lx + my = 1$ and find its equation.

7. The conics whose equations referred to rectangular axes are

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$
$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0,$$

intersect in four concyclic points; prove that $\frac{a - b}{h} = \frac{a' - b'}{h'}$ and that the coordinates of the centre of the circle are

$$\left(\frac{hg' - h'g}{ah' - a'h}, \quad \frac{hf' - h'f}{ah' - a'h}\right).$$

214. Equation of pair of tangents from a point.

We may make use of articles 211, 212 to find the equation of the pair of tangents from (x_1, y_1) to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)},$$

which is supposed not to be two lines.

The chord of contact of the tangents from (x_1, y_1) is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0 \ldots(2).$$

Now the pair of tangents from (x_1, y_1) to (1) is one of the conics having double contact with (1) at the points where

it is met by (2), and the general equation of such conics is (§ 212)

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - k\{axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c\}^2 = 0 \quad\text{.......................(3)}.$$

If now we choose k so that (3) will pass through the point (x_1, y_1) we shall have the equation of the pair of tangents.

We find that

$$k = \frac{1}{ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c}.$$

Thus the pair of tangents has for its equation

$$\begin{aligned}&(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)\\ &\quad\times(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)\\ &\quad= \{axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c\}^2,\end{aligned}$$

which is what we obtained long ago in § 142 and which we have been writing

$$SS_1 = T^2.$$

Foci of Conics.

215. We have seen that the equations of the central conics referred to their principal axes are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad\text{.......................(1)},$$

b^2 being negative if the conic be a hyperbola.

Now the conic (1) has two foci

$$(\sqrt{a^2 - b^2}, 0) \text{ and } (-\sqrt{a^2 - b^2}, 0),$$

and two corresponding directrices

$$x = \frac{a^2}{\sqrt{a^2 - b^2}} \text{ and } x = -\frac{a^2}{\sqrt{a^2 - b^2}}.$$

The symmetry of the equation (1) shews that there are yet two other foci whose coordinates are

$$(0, \sqrt{b^2 - a^2}) \text{ and } (0, -\sqrt{b^2 - a^2}),$$

and two corresponding directrices, viz.

$$y = \frac{b^2}{\sqrt{b^2 - a^2}} \quad \text{and} \quad y = -\frac{b^2}{\sqrt{b^2 - a^2}},$$

and clearly the conic might be regarded as the locus of points whose distance from either of the points

$$(0, \pm \sqrt{b^2 - a^2})$$

is a constant times their distance from the corresponding one of the two lines

$$y = \pm \frac{b^2}{\sqrt{b^2 - a^2}};$$

the constant, or eccentricity will now be $\sqrt{\frac{b^2 - a^2}{b^2}}$, just as before it was $\sqrt{\frac{a^2 - b^2}{a^2}}$.

Now we see that the two new foci and directrices are imaginary whether the conic be an ellipse or hyperbola. The new eccentricity is also imaginary if the conic be an ellipse but it is real if the conic be a hyperbola.

Central conics then have four foci, two real and lying on one axis, the other two imaginary and lying on the other axis. Of the corresponding eccentricities one is real and one imaginary in the case of an ellipse, and both are real if the conic be a hyperbola.

A parabola being the limiting case of an ellipse, we may say of it too that it has four foci but three of them are now 'at infinity.'

Example. Each directrix of a conic is the polar of the corresponding focus.

216. Equation of pair of tangents from a focus.

The equation of the pair of tangents from (x_1, y_1) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

Therefore the equation of the pair of tangents from the focus $(\sqrt{a^2-b^2}, 0)$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2-b^2}{a^2}-1\right)=\left(\frac{x\sqrt{a^2-b^2}}{a^2}-1\right)^2,$$

that is

$$\frac{b^2}{a^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\frac{a^2-b^2}{a^4}x^2+1-\frac{2x\sqrt{a^2-b^2}}{a^2}=0,$$

that is

$$x^2-2x\sqrt{a^2-b^2}+a^2-b^2+y^2=0,$$

which can be written

$$(x-\sqrt{a^2-b^2})^2+y^2=0.$$

Similarly the pair of tangents from the focus $(-\sqrt{a^2-b^2}, 0)$ is

$$(x+\sqrt{a^2-b^2})^2+y^2=0,$$

and the pair of tangents from the two imaginary foci can be written

$$x^2+(y-\sqrt{b^2-a^2})^2=0$$

and

$$x^2+(y+\sqrt{b^2-a^2})^2=0.$$

Thus the equation of the pair of tangents from a focus is that of a 'point circle' at the focus.

Hence we get the following result:

If (x_1, y_1) be a focus of a conic whose Cartesian equation is given, referred to rectangular axes, then the equation of the pair of tangents from (x_1, y_1) to the conic is

$$(x-x_1)^2+(y-y_1)^2=0.$$

If the axes be oblique the pair of tangents will be

$$(x-x_1)^2+(y-y_1)^2+2(x-x_1)(y-y_1)\cos\omega=0.$$

217. *To find the foci of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the axes being rectangular.

Let (x_1, y_1) be a focus.

The equation of the pair of tangents from (x_1, y_1) is

$$(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) \times (ax^2 + 2hxy + by^2 + 2gx + 2fy + c) - \{(ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + (gx_1 + fy_1 + c)\}^2 = 0.$$

This then must reduce to a 'point circle' at (x_1, y_1),

$\therefore$ coefficient of x^2 = coefficient of y^2,

and coefficient of $xy = 0$.

$$\therefore (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)(a - b) - (ax_1 + hy_1 + g)^2 + (hx_1 + by_1 + f)^2 = 0,$$

and

$$h(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) - (ax_1 + hy_1 + g)(hx_1 + by_1 + f) = 0.$$

Thus the coordinates of the foci are given by the equations

$$\frac{(ax + hy + g)^2 - (hx + by + f)^2}{a - b} = \frac{(ax + hy + g)(hx + by + f)}{h}$$
$$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$

Another method for finding the foci of conics will be given in a later chapter when we come to deal with 'tangential equations.'

218. Equation of the axes.

It follows from the preceding paragraph that the equation of the axes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$\frac{\xi^2 - \eta^2}{a - b} = \frac{\xi\eta}{h} \quad \text{.........................(1)},$$

where $\xi \equiv ax + hy + g, \quad \eta = hx + by + f.$

For (1) is a conic, and the foci lie upon it (§ 217). Moreover it is satisfied by $\xi = 0$, $\eta = 0$, that is by the centre of the conic.

Hence (1) is a conic through the four foci and the centre of the conic; but the axes of the conic pass through these five points, and there can be only one conic through five points of which not more than three are collinear (§ 210).

Thus (1) represents the axes of the conic.

This equation of the axes is extremely easy to remember on account of its resemblance to the equation of the bisectors of the angles between two lines (§ 61).

Examples. 1. Find equations to give the foci of the conic given by the general equation when the axes are inclined at an angle ω and shew that the equation of the axes is

$$\begin{vmatrix} a, & 1, & \xi^2 \\ b, & 1, & \eta^2 \\ h, & \cos\omega, & \xi\eta \end{vmatrix} = 0.$$

2. Obtain the coordinates of the foci of the ellipse

$$8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0,$$

and shew that the equations of its axes are

$$2x - y - 1 = 0 \text{ and } 2x + 4y - 11 = 0.$$

219. We may also obtain the equation of the axes of the conic from the fact that the axes are the bisectors of the angles between the asymptotes.

Now the asymptotes are parallel to the lines (§ 189)

$$ax^2 + 2hxy + by^2 = 0.$$

Thus the axes must be the lines through the centre of the conic parallel to the lines

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.$$

That is, the equation of the axes is

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h},$$

where (x_1, y_1) are the coordinates of the centre, that is to say x_1 and y_1 are given by

$$ax_1 + hy_1 + g = 0,$$

$$hx_1 + by_1 + f = 0.$$

220. Lengths and position of the axes.

If it be required to find the lengths as well as the positions of the axes of the conic given by the general equation we begin by transferring the origin to the centre.

If (x_1, y_1) be the centre

$$ax_1 + hy_1 + g = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$hx_1 + by_1 + f = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

On transferring the origin to (x_1, y_1) the equation of the conic becomes

$$ax^2 + 2hxy + by^2 + c' = 0,$$

where $$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \equiv c' \ldots\ldots\ldots(3).$$

Now $(3) - (1) \times x_1 - (2) \times y_1$ gives

$$gx_1 + fy_1 + c - c' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating x_1 and y_1 from (1), (2) and (4) we get

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c - c' \end{vmatrix} = 0,$$

$$\therefore \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} + \begin{vmatrix} a, & h, & 0 \\ h, & b, & 0 \\ g, & f, & -c' \end{vmatrix} = 0,$$

that is $\Delta - c'C = 0$ where $C \equiv ab - h^2$,

$$\therefore \; c' = \frac{\Delta}{C}.$$

The equation of the conic is now

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0 \quad \ldots\ldots\ldots\ldots\ldots(5).$$

It becomes now a matter of finding the lengths of the axes of a conic whose equation is of the *form*

$$ax^2 + 2hxy + by^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

since our equation (5) reduces to this form when we divide by

$$-\frac{\Delta}{C}.$$

221. *To find the lengths and position of the axes of the conic whose equation is*

$$ax^2 + 2hxy + by^2 = 1 \quad \text{..................(1).}$$

The procedure is very much the same whether the axes of coordinates be rectangular or oblique; but it may seem easier to the student to take the case where the axes are rectangular first.

Consider the circle of radius r with its centre at the centre of the conic. Its equation is

$$\frac{x^2 + y^2}{r^2} = 1 \quad \text{..........................(2).}$$

Subtracting (2) from (1) we have an equation

$$\left(a - \frac{1}{r^2}\right) x^2 + 2hxy + \left(b - \frac{1}{r^2}\right) y^2 = 0 \quad \text{.........(3),}$$

which represents a pair of lines through the origin and the intersection of (1) and (2).

These straight lines will become coincident when and only when they lie along the axes of the conic.

The condition that the lines should be coincident is

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2 \quad \text{..................(4).}$$

This is a quadratic equation in r^2. If the conic be an ellipse both the values of r^2 will be positive; but if it be a hyperbola one will be positive (giving the transverse axis) and the other negative (giving the conjugate axis). If r_1^2, r_2^2 be the roots and both be positive the conic is an ellipse with $2r_1$, $2r_2$ for its axes. But if r_1^2 be positive and r_2^2 negative the conic is a hyperbola the length of whose transverse axis is $2r_1$ and the length of the conjugate axis $2\sqrt{-r_2^2}$ or $2ir_2$, where i is $\sqrt{-1}$.

The actual positions of the two axes are easily found.

15—2

For r_1^2 being one of the values of r^2 given by (4) the lines (3) are

$$\left[\left(a-\frac{1}{r_1^2}\right)x+hy\right]^2=0,$$

that is the pair of coincident lines

$$\left(a-\frac{1}{r_1^2}\right)x+hy=0.$$

This then is the equation of the axis corresponding to r_1, and the equation of the other axis will be

$$\left(a-\frac{1}{r_2^2}\right)x+hy=0.$$

222. If the axes of coordinates be oblique, the equation of the circle of radius r having its centre at the centre of the conic will be

$$\frac{x^2+2xy\cos\omega+y^2}{r^2}=1,$$

so that the equation of the pair of lines through the centre of the conic and the points of intersection of these lines with the conic will be

$$\left(a-\frac{1}{r^2}\right)x^2+2\left(h-\frac{\cos\omega}{r^2}\right)xy+\left(b-\frac{1}{r^2}\right)y^2=0.$$

The equation giving the lengths of the semi-axes will now be

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=\left(h-\frac{\cos\omega}{r^2}\right)^2,$$

which can be written

$$\frac{1}{r^4}-\left(\frac{a+b-2h\cos\omega}{\sin^2\omega}\right)\frac{1}{r^2}+\frac{ab-h^2}{\sin^2\omega}=0.$$

If r_1^2 and r_2^2 be the roots the equations of the corresponding axes will be

$$\left(a-\frac{1}{r_1^2}\right)x+\left(h-\frac{\cos\omega}{r_1^2}\right)y=0,$$

$$\left(a-\frac{1}{r_2^2}\right)x+\left(h-\frac{\cos\omega}{r_2^2}\right)y=0.$$

223. The advantage of the method adopted in the preceding articles is that the equation of each axis is given separately and associated with the length of that particular axis, so that we are able to say which is the major axis and which the minor, or which is the transverse and which the conjugate axis.

Except for this, however, it is much simpler to get the *equation* of the axes as we have done in § 218 and to find the *lengths* by means of invariants.

Thus if $$ax^2 + 2hxy + by^2 = 1$$
become on transformation to principal axes
$$\frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} = 1,$$
we have that
$$ax^2 + 2hxy + by^2 = \frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2}.$$

Thus by invariants
$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} = \frac{\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} - 0}{\sin^2\dfrac{\pi}{2}} = \frac{1}{r_1^2} + \frac{1}{r_2^2}$$
and
$$\frac{ab - h^2}{\sin^2\omega} = \frac{1}{r_1^2 r_2^2}.$$

Hence $\dfrac{1}{r_1^2}$ and $\dfrac{1}{r_2^2}$ are the roots of the equation in $\dfrac{1}{r^2}$, viz.
$$\frac{1}{r^4} - \left(\frac{a + b - 2h\cos\omega}{\sin^2\omega}\right)\frac{1}{r^2} + \frac{ab - h^2}{\sin^2\omega} = 0.$$

Examples. 1. Find the positions and lengths of the axes of the conic
$$5x^2 - 6xy + 5y^2 + 22x - 26y + 29 = 0,$$
the axes being rectangular.

[The centre is given by
$$5x - 3y + 11 = 0,$$
$$-3x + 5y - 13 = 0,$$
from which we find $x = -1$, $y = 2$.

On transferring the origin to this point we find that the equation of the conic becomes

$$5x^2-6xy+5y^2-8=0,$$

that is
$$\frac{5}{8}x^2-2\left(\frac{3}{8}\right)xy+\frac{5}{8}y^2=1,$$

so that
$$a=\frac{5}{8},\quad h=-\frac{3}{8},\quad b=\frac{5}{8}.$$

The lengths of the semi-axes are then given by

$$\left(\frac{5}{8}-\frac{1}{r^2}\right)\left(\frac{5}{8}-\frac{1}{r^2}\right)=\frac{9}{64},$$

$$\therefore\ \frac{5}{8}-\frac{1}{r^2}=\pm\frac{3}{8},$$

$$\therefore\ \frac{1}{r^2}=\frac{5\pm3}{8},\quad \therefore\ r^2=4 \text{ or } 1.$$

The major axis is then 4 and the minor axis 2.

The equation of the line of the major axis is

$$\left(\frac{5}{8}-\frac{1}{4}\right)x-\frac{3}{8}y=0,$$

i.e. $x-y=0$.

The equation of the line of the minor axis is

$$\left(\frac{5}{8}-1\right)x-\frac{3}{8}y=0,$$

i.e. $x+y=0$.

These are of course the equations of the axes of the ellipse referred to the new axes of coordinates. The equation of the major axis referred to the original axes will be

$$(x+1)-(y-2)=0,\ \text{that is}\ x-y+3=0,$$

and of the minor axis

$$(x+1)+(y-2)=0,\ \text{that is}\ x+y-1=0.]$$

2. Find the positions and lengths of the axes of the conics

(i) $9x^2+4xy+6y^2-22x-16y+9=0$,

(ii) $7x^2+12xy-2y^2-26x-8y+7=0$,

the axes being rectangular, and represent the same in a figure.

224. Eccentricity of Central Conic.

Let e be an eccentricity of the conic

$$ax^2+2hxy+by^2=1$$

referred to rectangular axes.

Let this become on transformation to principal axes

$$a'x'^2 + b'y'^2 = 1,$$

$$\therefore \frac{1}{b'} = \frac{1}{a'}(1 - e^2),$$

that is $$a' = b'(1 - e^2).$$

Also $$a' + b' = a + b,$$

$$a'b' = ab - h^2.$$

Eliminating a' and b' we get the equation

$$e^4 + \frac{(a-b)^2 + 4h^2}{ab - h^2}(e^2 - 1) = 0.$$

In the case where the conic is an ellipse, i.e. $ab - h^2 > 0$, the two values of e^2 given by this have opposite sign. The positive value of e^2 then gives the real eccentricity.

But if the conic be a hyperbola, i.e. $ab - h^2 < 0$, both the values of e^2 given by this equation are positive and the question arises which belongs to a real focus and directrix and which to an imaginary one. We shall investigate this point further in the next article.

225. Let the conic

$$ax^2 + 2hxy + by^2 = 1$$

on transformation to principal axes be

$$\frac{x'^2}{\alpha^2} + \frac{y'^2}{\beta^2} = 1,$$

and let β^2 be negative if either α^2 or β^2 is negative, and let $\alpha^2 > \beta^2$ if both be positive.

Let e be the eccentricity associated with a real focus and directrix.

Then $$e^2 = \frac{\alpha^2 - \beta^2}{\alpha^2}.$$

Using invariants we have

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=a+b,$$

$$\frac{1}{\alpha^2\beta^2}=ab-h^2.$$

Whence

$$\alpha^2+\beta^2=\frac{a+b}{ab-h^2} \quad \text{and} \quad \alpha^2\beta^2=\frac{1}{ab-h^2},$$

$$\therefore \ \alpha^2-\beta^2=\sqrt{\frac{(a+b)^2-4(ab-h^2)}{(ab-h^2)^2}},$$

with a positive sign and

$$\frac{1}{\alpha^2}-\frac{1}{\beta^2}=\mp\sqrt{(a+b)^2-4(ab-h^2)},$$

according as the conic is an ellipse or a hyperbola.

Whence $\dfrac{2}{\alpha^2}=a+b\mp\sqrt{(a+b)^2-4(ab-h^2)},$

according as the conic is an ellipse or a hyperbola.

$$\therefore \ e^2=\tfrac{1}{2}\sqrt{\frac{(a+b)^2-4(ab-h^2)}{(ab-h^2)^2}}$$

$$\times\{(a+b)\mp\sqrt{(a+b)^2-4(ab-h^2)}\}$$

according as the conic is an ellipse or a hyperbola, where the positive sign is taken with the radical outside the bracket. That is, if the conic be an ellipse,

$$e^2=\frac{\sqrt{(a-b)^2+4h^2}}{2(ab-h^2)}\{(a+b)-\sqrt{(a-b)^2+4h^2}\}.$$

But if the conic be a hyperbola

$$e^2=\frac{\sqrt{(a-b)^2+4h^2}}{2(h^2-ab)}\{(a+b)+\sqrt{(a-b)^2+4h^2}\}$$

$$=\frac{(a-b)^2+4h^2+(a+b)\sqrt{(a-b)^2+4h^2}}{2(h^2-ab)}.$$

This gives the eccentricity of the hyperbola associated with a real focus and directrix.

The eccentricity of the hyperbola associated with the imaginary focus and directrix will be given by the other root of the quadratic equation in e^2 found in the last article, and this will be got by simply changing the signs of the radical, that is,

$$e^2 = \frac{-\sqrt{(a-b)^2+4h^2}}{2(h^2-ab)}\{a+b-\sqrt{(a-b)^2+4h^2}\}$$

$$= \frac{(a-b)^2+4h^2-(a+b)\sqrt{(a-b)^2+4h^2}}{2(h^2-ab)}.$$

It may here be remarked that the eccentricity of a hyperbola associated with an imaginary focus and directrix is the same as the eccentricity of the conjugate hyperbola associated with a real focus and directrix.

So that the second value of e^2 just found gives the eccentricity of the conjugate hyperbola.

We may observe that if $a+b=0$, that is if the conic be a rectangular hyperbola, both of the values of e^2 become $\frac{4(h^2-ab)}{2(h^2-ab)}$, that is $e=\sqrt{2}$, which is correct for a rectangular hyperbola.

Example. If the axes of coordinates be not rectangular the eccentricity e of the conic

$$ax^2+2hxy+by^2=1$$

is given by

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin^2\omega}.$$

226. Director circle.

To find the director circle of the central conic whose equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the axes being rectangular.

The director circle is the locus of the points of intersection of pairs of tangents at right angles. Now the equation of the pair of tangents from (x_1, y_1) is

$$(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)$$
$$-\{(ax_1+hy_1+g)x+(hx_1+by_1+f)y+(gx_1+fy_1+c)\}^2=0.$$

These tangents will be at right angles if

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0,$$

that is, if

$$(a+b)(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) - (ax_1 + hy_1 + g)^2 - (hx_1 + by_1 + f)^2 = 0,$$

that is if

$$(ab - h^2)(x_1^2 + y_1^2) - 2(hf - bg)x_1 - 2(gh - af)y_1 + (a+b)c - g^2 - f^2 = 0.$$

Thus the locus of (x_1, y_1) which is the director circle of the conic is

$$(ab - h^2)(x^2 + y^2) - 2(hf - bg)x - 2(gh - af)y + bc - f^2 + ca - g^2 = 0,$$

which may be written

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0,$$

where the capital letters are, as in § 133, the minors taken with their proper signs of the corresponding small letters in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

that is to say,

$$A = \begin{vmatrix} b, & f \\ f, & c \end{vmatrix} = bc - f^2, \quad B = \begin{vmatrix} a, & g \\ g, & c \end{vmatrix} = ac - g^2,$$

$$C = \begin{vmatrix} a, & h \\ h, & b \end{vmatrix} = ab - h^2,$$

$$F = -\begin{vmatrix} a, & h \\ g, & f \end{vmatrix} = gh - af, \quad G = \begin{vmatrix} h, & b \\ g, & f \end{vmatrix} = hf - bg,$$

$$H = -\begin{vmatrix} h, & f \\ g, & c \end{vmatrix} = fg - ch.$$

COR. *The equation of the directrix of the parabola represented by the general equation is*

$$2Gx + 2Fy - (A + B) = 0.$$

For the directrix of the parabola is the locus of the points of intersection of pairs of tangents at right angles, and $C \equiv ab - h^2 = 0$ when the conic is a parabola.

We can at once find the coordinates of the focus of this parabola. For if (x_1, y_1) be the focus, the equation of the directrix, which is the polar of the focus, must be

$$(ax_1 + hy_1 + g)\,x + (hx_1 + by_1 + f)\,y + (gx_1 + fy_1 + c) = 0.$$

This then must be identical with the equation for the directrix given above,

$$\therefore \quad \frac{ax_1 + hy_1 + g}{2G} = \frac{hx_1 + by_1 + f}{2F} = \frac{gx_1 + fy_1 + c}{-(A + B)}.$$

These two equations determine x_1 and y_1.

Example. Find the equation of the directrix and the coordinates of the focus of the parabola

$$x^2 + 2xy + y^2 - 3x + 6y - 4 = 0.$$

227. Proposition. *The four directrices of a central conic pass through the points common to the conic and its director circle.*

For if the conic be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{........................(1)},$$

the director circle is

$$x^2 + y^2 = a^2 + b^2 \quad \text{.....................(2)}.$$

The general equation of conics through the points of intersection of these two is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + k\,(x^2 + y^2 - \overline{a^2 + b^2}) = 0.$$

Choose k so that the term in y^2 disappears.

This requires that $\qquad k = -\dfrac{1}{b^2}.$

We thus get as one of the conics through the points of intersection of (1) and (2)

$$\frac{x^2}{a^2} - 1 - \frac{1}{b^2}(x^2 - a^2 - b^2) = 0,$$

that is $$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \frac{a^2}{b^2} = 0,$$

that is $$x^2 - \frac{a^4}{a^2 - b^2} = 0,$$

which is the two directrices

$$x = \pm \frac{a^2}{\sqrt{a^2 - b^2}}.$$

Similarly the other two directrices, viz.

$$y^2 - \frac{b^4}{b^2 - a^2} = 0,$$

are a conic through the points of intersection of (1) and (2).

Thus the proposition is proved.

Example. Shew that the equation of a pair of directrices of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is of the form

$$(ax + hy + g)^2 + (hx + by + f)^2 + kS = 0.$$

228. A theorem of Newton's.

The following proposition due to Newton which is usually proved in works on Pure Geometry is extremely easy to establish by Analytical methods.

If O be a variable point in the plane of a conic and lines OPQ, ORS be drawn through O in fixed directions to cut the conic in P, Q, R, S, then the ratio $OP \,.\, OQ : OR \,.\, OS$ is constant.

Refer the conic to the lines OPQ and ORS as axes. Its equation will be of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Putting $y = 0$ we get

$$ax^2 + 2gx + c = 0.$$

Thus $OP \,.\, OQ =$ product of the roots of this equation $= \dfrac{c}{a}$.

Similarly $OR \,.\, OS = \dfrac{c}{b}$,

$\therefore\ OP \,.\, OQ : OR \,.\, OS = \dfrac{1}{a} : \dfrac{1}{b} = b : a.$

Now if we transfer the origin to any other point O', *keeping the directions of the axes unchanged*, the coefficients of the terms of the highest order in the equation are unchanged. Thus the new equation will be of the form

$$ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0,$$

and if the new axes cut the conic in P', Q' and R', S' respectively,

$$O'P' \,.\, O'Q' : O'R' \,.\, O'S' = b : a.$$

Thus $\quad OP \,.\, OQ : OR \,.\, OS = O'P' \,.\, O'Q' : O'R' \,.\, O'S'.$

And the proposition is proved.

229. It must be understood that the distances OP, OQ, &c. in the preceding paragraph are algebraical. That is to say, if OP and OQ are in opposite directions they have opposite sign, or in other words $OP \,.\, OQ$ is a negative quantity.

The well known property of the ellipse that, if QV be an ordinate of a diameter PCP' and DCD' be the diameter conjugate to CP,

$$QV^2 : PV \,.\, VP' = CD^2 : CP^2$$

is a special case of the theorem just proved.

For if QV meet the ellipse again in Q' we have

$$VQ \,.\, VQ' : VP \,.\, VP' = CD \,.\, CD' : CP \,.\, CP',$$

that is $\quad -QV^2 : -PV \,.\, VP' = -CD^2 : -CP^2,$

or $\quad QV^2 : PV \,.\, VP' = CD^2 : CP^2.$

The same property is true for the hyperbola, but it must be understood that D and D' are the points where the conjugate diameter meets the hyperbola itself, and not where it meets the conjugate hyperbola.

But if it meets the conjugate hyperbola in d and d', then

$$Cd^2 = -CD^2.$$

So that we have

$$QV^2 : PV \,.\, VP' = -Cd^2 : CP^2,$$

that is
$$QV^2 : PV \,.\, P'V = Cd^2 : CP^2,$$

which is the geometrical property with which the student is familiar.

Example. If lines OPQ, ORS be drawn through a point O in the plane of a conic, prove that the ratio $OP \,.\, OQ : OR \,.\, OS$ is equal to the ratio of the squares of the diameters parallel to OPQ and ORS respectively.

230. Contact of conics.

We have already in § 209 said something of the different degrees of the contact of conics, so that it is already understood what is meant by simple or two point contact, three point contact, and four point contact.

Suppose that (x_1, y_1) be a point on the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{......(1)},$$

then

$$T \equiv axx_1 + h\,(xy_1 + x_1y) + byy_1 + g\,(x + x_1) + f(y + y_1) + c = 0$$

is the tangent at (x_1, y_1).

The general equation of conics having simple contact with (1) at (x_1, y_1) will be

$$S + T\,(lx + my + n) = 0 \quad \text{..................(2)},$$

for this is a conic which has as common chords with the given conic the tangent $T = 0$ and a line

$$lx + my + n = 0.$$

If the line $lx + my + n = 0$ passes through (x_1, y_1) the conics will have three point contact at (x_1, y_1). The general equation of conics having three point contact with (1) at (x_1, y_1) is then

$$S + T\,\{l\,(x - x_1) + m\,(y - y_1)\} = 0 \quad \text{...........(3)}.$$

And in the special case where the line $lx + my + n = 0$ becomes the tangent, the conics will have four points common

at (x_1, y_1), in other words they will have four point contact. Thus the general equation of conics having four point contact at (x_1, y_1) with the given conic is

$$S + kT^2 = 0.$$

Examples. 1. Shew that the equation of the parabola having four point contact with the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point P $(a\cos\theta, b\sin\theta)$ is

$$\left(\frac{x}{a}\sin\theta - \frac{y}{b}\cos\theta\right)^2 + \frac{2x\cos\theta}{a} + \frac{2y\sin\theta}{b} - 2 = 0,$$

and prove that its latus rectum is $\dfrac{2a^2b^2}{CP^3}$.

2. Find the equation of the rectangular hyperbola having four point contact with the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a\cos\theta, b\sin\theta)$.

231. Circles of curvature.

If two curves have three point contact at a point, they are said to have the same curvature here. By the curvature of a curve is meant the rate at which the tangent is deflecting. Now when two curves have three point contact they may be regarded as the limiting case of two curves having three very near points in common. But two curves with three consecutive points in common will have two consecutive tangents in common. And thus they will have the same curvature, as the tangent is deflecting at the same rate for them both.

The circle which has three point contact with a curve at a point is called the 'circle of curvature' at that point. It is also called the 'osculating circle' at that point.

The measure of the curvature of a circle at any point of it is $\dfrac{1}{r}$ where r is the radius. For if P be a point on the circle and if the tangent at any other point Q meet that at P in T, the deflection of the tangent between P and Q, that is to say the angle between the tangents at P and Q, is equal to the angle between the radii OP and OQ.

The rate of deflection per unit length of arc is of course constant for a circle so that it will be

$$\frac{\angle POQ}{\text{arc } PQ} = \frac{\text{arc } PQ}{r} \div \text{arc } PQ = \frac{1}{r}.$$

Thus if ρ be the radius of the circle of curvature at a point P of a curve, we say that its curvature there is $\frac{1}{\rho}$.

It is important to distinguish the 'curvature at a point,' and what is called the 'radius of curvature at a point.' By the radius of curvature at a point is meant the radius of the circle of curvature at the point. By the curvature is meant the reciprocal of the radius of this circle.

The 'centre of curvature' at a point of a curve means the centre of the circle of curvature at that point.

232. Equation of circle of curvature.

The equation of the circle of curvature at a point (x_1, y_1) of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots(1)$$

is very easily obtained.

For the circle of curvature is one of the conics having three point contact with (1) at (x_1, y_1), these having their equations included in

$$S + T\{l(x - x_1) + m(y - y_1)\} = 0 \quad \ldots\ldots\ldots(2),$$

as in § 230.

We have then only to choose l and m so that (2) shall be a circle and it will then be the circle of curvature required.

233. *To find the circle of curvature of the parabola $y^2 = 4ax$ at the point $P(a\mu^2, 2a\mu)$.*

The tangent at P is $y - \frac{x}{\mu} - a\mu = 0$.

Therefore the equation of the common chord PQ of the parabola and its circle of curvature at P will be (§ 213)

$$y + \frac{x}{\mu} = 2a\mu + \frac{a\mu^2}{\mu} = 3a\mu.$$

Thus the circle of curvature will be obtained from

$$y^2 - 4ax - k\left(y - \frac{x}{\mu} - a\mu\right)\left(y + \frac{x}{\mu} - 3a\mu\right) = 0$$

by making the coefficient of $x^2 =$ that of y^2.

This will give $1 - k = \frac{k}{\mu^2}$, $\therefore k = \frac{\mu^2}{\mu^2 + 1}$.

The equation of the circle of curvature is then

$$(\mu^2 + 1)(y^2 - 4ax) - \mu^2\left(y^2 - \frac{x^2}{\mu^2} - 4a\mu y + 2ax + 3a^2\mu^2\right) = 0,$$

which is

$$x^2 + y^2 - 2ax(2 + 3\mu^2) + 4a\mu^3 y = 3a^2\mu^4,$$

that is

$$\begin{aligned}\{x - a(2 + 3\mu^2)\}^2 + (y + 2a\mu^3)^2 &= a^2\{(2 + 3\mu^2)^2 + 4\mu^6 + 3\mu^4\}\\ &= a^2(4 + 12\mu^2 + 12\mu^4 + 4\mu^6)\\ &= 4a^2(1 + \mu^2)^3.\end{aligned}$$

From which we see that the coordinates of the centre of curvature are

$$\{a(2 + 3\mu^2), \quad -2a\mu^3\},$$

and the radius of curvature is $2a(1 + \mu^2)^{\frac{3}{2}} = 2a \operatorname{cosec}^3 \theta$, where θ is the angle which the tangent at P makes with the axis of the parabola.

234. *To find the circle of curvature at the point P $(a\cos\theta, b\sin\theta)$ of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The tangent at P is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

Thus the common chord of the ellipse and the circle of curvature will be

$$\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

The circle of curvature will then be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - k\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1\right)\left(\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta - \cos 2\theta\right) = 0,$$

k being so chosen that the coefficient of $x^2 =$ that of y^2.

We have then

$$\frac{1}{a^2}-\frac{k\cos^2\theta}{a^2}=\frac{1}{b^2}+\frac{k\sin^2\theta}{b^2}=\lambda \text{ (say)},$$

so that
$$k\cos^2\theta=1-a^2\lambda,$$
$$k\sin^2\theta=-1+b^2\lambda,$$
$$\therefore k=-(a^2-b^2)\lambda \quad\text{and}\quad a^2\lambda=1+(a^2-b^2)\lambda\cos^2\theta,$$
$$\therefore \lambda=\frac{1}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

The equation of the circle of curvature is

$$\lambda(x^2+y^2)+\frac{kx}{a}\cos\theta(1+\cos 2\theta)-\frac{ky}{b}\sin\theta(1-\cos 2\theta)$$
$$=1+k\cos 2\theta,$$

that is

$$x^2+y^2-\frac{2(a^2-b^2)x}{a}\cos^3\theta+\frac{2(a^2-b^2)y}{b}\sin^3\theta$$
$$=a^2\sin^2\theta+b^2\cos^2\theta-(a^2-b^2)\cos 2\theta,$$

which reduces to

$$\left(x-\frac{a^2-b^2}{a}\cos^3\theta\right)^2+\left(y+\frac{a^2-b^2}{b}\sin^3\theta\right)^2=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^3}{a^2b^2},$$

so that the centre of curvature is

$$\left(\frac{a^2-b^2}{a}\cos^3\theta,\quad \frac{b^2-a^2}{b}\sin^3\theta\right)$$

and the radius of curvature is $\dfrac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{ab}$, which equals $\dfrac{CD^3}{ab}$ where CD is the semi-diameter conjugate to CP.

235. Simplification by Differential method. The method of finding the centre and radius of curvature at a point of a parabola or of an ellipse is greatly simplified if we use the methods of the differential calculus.

The centre of curvature at a point is the intersection of the normal at the point with the normal at a consecutive point, for the normals at two points of a circle meet in the centre

of the circle, and the curve has two consecutive normals in common with the circle.

Thus to find the centre of curvature at the point $(a\mu^2, 2a\mu)$ of the parabola $y^2 = 4ax$ we proceed thus:

The normal is $\qquad y + \mu x = 2a\mu + a\mu^3.$

Where this meets the consecutive normal

$$x = 2a + 3a\mu^2,$$

this being obtained by differentiating with respect to μ.

We thus find as the coordinates of the centre of curvature

$$(2a + 3a\mu^2, \quad -2a\mu^3).$$

So for the ellipse, the normal at θ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2,$$

or $$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2.$$

To find where this meets the consecutive normal we differentiate with respect to θ and obtain

$$ax\sec\theta\tan\theta + by\operatorname{cosec}\theta\cot\theta = 0.$$

On solving for x and y we get

$$x = \frac{(a^2 - b^2)}{a}\cos^3\theta, \quad y = -\frac{(a^2 - b^2)}{b}\sin^3\theta.$$

The radius of curvature can be obtained by finding the distance between the centre of curvature and the point $(a\cos\theta, b\sin\theta)$. Thus if ρ be the radius of curvature,

$$\rho^2 = \left(a\cos\theta - \frac{a^2 - b^2}{a}\cos^3\theta\right)^2 + \left(b\sin\theta + \frac{a^2 + b^2}{b}\sin^3\theta\right)^2$$
$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^3}{a^2b^2}.$$

Or we may use the formula of the differential calculus

$$\rho = \frac{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{\frac{dx}{d\theta}\cdot\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}}{ab},$$

for $x = a\cos\theta$, $y = b\sin\theta$.

236. Intersection of ellipse and circle of curvature.

If the circle of curvature of an ellipse at a point P whose eccentric angle is θ cut the ellipse in Q, whose eccentric angle is ϕ, then the circle and ellipse meet in the four points whose eccentric angles are

$$\theta, \quad \theta, \quad \theta, \quad \phi.$$

Therefore by § 213

$$3\theta + \phi = \text{an even multiple of } \pi.$$

Thus the coordinates of Q will be $(a \cos 3\theta, -b \sin 3\theta)$ and the eccentric angle of Q may be taken as -3θ.

As the chord through θ and ϕ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a} \cos \frac{\theta + \phi}{2} + \frac{y}{b} \sin \frac{\theta + \phi}{2} = \cos \frac{\theta - \phi}{2},$$

we get at once that the equation of PQ is

$$\frac{x}{a} \cos \theta - \frac{y}{b} \sin \theta = \cos 2\theta,$$

which agrees with what we obtained before.

We may observe that from the above equation

$$3\theta + \phi = 2n\pi,$$

which gives $$\theta = \tfrac{1}{3}(2n\pi - \phi),$$

it follows that the circles of curvature at the points whose eccentric angles are $\frac{1}{3}(2\pi - \phi)$, $\frac{1}{3}(4\pi - \phi)$, $\frac{1}{3}(6\pi - \phi)$ will all cut the ellipse at the same point ϕ.

Examples. 1. The circle of curvature of the parabola $y^2 = 4ax$ at the point $(a\mu^2, 2a\mu)$ will meet the curve again in the point $(9a\mu^2, -6a\mu)$. [See Ex. 2, § 213.]

2. The locus of the middle point of the common chord of a parabola and its circle of curvature at any point is a parabola whose latus rectum is one-fifth that of the given parabola.

3. The circles of curvature at three points P, Q, R of an ellipse all cut the ellipse again in the same point, prove that the centre of the ellipse is the centre of mean position of P, Q, R.

4. Prove that the circles of curvature at the vertices of a conic have four point contact with the conic.

237. Conic referred to tangent and normal at a point.

It is sometimes convenient to take as the axes of coordinates the tangent and normal at a point of a conic. Let us take the axis of x to be the tangent at a point P of the conic, and let the normal be the axis of y.

As the origin is on the curve, the constant term in the equation disappears and the equation of the conic is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$$

To find where the axis of x meets the conic put $y = 0$. We get then $$ax^2 + 2gx = 0.$$

Both the roots of this equation have to be zero, therefore $g = 0$.

Thus the equation of the conic is of the form

$$ax^2 + 2hxy + by^2 + 2fy = 0.$$

This is still the form of the equation of the conic when we take a tangent to the curve to be the axis of x and any line through its point of contact to be the axis of y.

Examples. 1. All chords of a conic which subtend a right angle at a fixed point O on the conic cut the normal at O in a fixed point.

2. If $ax^2+2hxy+by^2+2fy=0$ be the equation of a conic referred to the tangent and normal at a point of it, the radius of curvature at the origin is $\dfrac{f}{a}$.

238. Conic referred to two tangents as axes.

The equation of a conic referred to two tangents as axes of coordinates is frequently of use. It is very easily obtained. For if the conic touch the tangents at distances h and k from the origin, it will be a conic having double contact with the axes, which form the conic $xy = 0$, at the points where they are met by the line $\dfrac{x}{h} + \dfrac{y}{k} - 1 = 0$.

Thus the equation of the conic will be of the form

$$xy + A\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 = 0,$$

or changing the constant A we may write this

$$\frac{x}{h}+\frac{y}{k}-1=\lambda\sqrt{\frac{xy}{hk}} \quad\text{.....................(1)},$$

which is also of the form

$$\frac{x^2}{h^2}+2\mu xy+\frac{y^2}{k^2}-\frac{2x}{h}-\frac{2y}{k}+1=0 \quad\text{............(2)}.$$

Either of these equations (1) and (2) represents for various values of λ and μ a system of conics touching the axes of x and y at distances h and k respectively from the origin.

In the particular case where the conic (2) is a parabola

$$\mu^2=\frac{1}{h^2k^2}, \text{ that is } \mu=\pm\frac{1}{hk}.$$

If we take $\mu=+\frac{1}{hk}$ the equation becomes

$$\left(\frac{x}{h}+\frac{y}{k}-1\right)^2=0,$$

which represents two coincident straight lines.

If we take $\mu=-\frac{1}{hk}$ we get a non-degenerate parabola whose equation is

$$\frac{x^2}{h^2}-\frac{2xy}{hk}+\frac{y^2}{k^2}-\frac{2x}{h}-\frac{2y}{k}+1=0,$$

which can be expressed in the irrational form

$$\sqrt{\frac{x}{h}}+\sqrt{\frac{y}{k}}-1=0 \quad\text{.....................(3)},$$

or, if we write α and β for $\frac{1}{h}$ and $\frac{1}{k}$,

$$\sqrt{\alpha x}+\sqrt{\beta y}-1=0.$$

Examples. 1. Shew that the tangent to the parabola

$$\sqrt{\frac{x}{h}}+\sqrt{\frac{y}{k}}=1$$

at the point (x_1, y_1) is

$$\frac{x}{\sqrt{hx_1}}+\frac{y}{\sqrt{ky_1}}=1.$$

2. The line $lx+my-1=0$ will touch the parabola $\sqrt{\alpha x}+\sqrt{\beta y}=1$ if

$$\frac{\alpha}{l}+\frac{\beta}{m}=1.$$

239. Cartesian representation of a system of conics through four points.

In general it is best to represent conics through four given points by means of 'homogeneous coordinates,' which are to be considered in later chapters. If at any time a representation of such conics in Cartesian coordinates is wanted, it can be obtained by taking the axes of coordinates to pass each through two of the given points. We may thus take the coordinates of the given points A, B, C, D to be

$$(h, 0), \quad (0, k), \quad (h', 0), \quad (0, k').$$

The equation of AB will be $\frac{x}{h}+\frac{y}{k}-1=0$, and that of CD will be $\frac{x}{h'}+\frac{y}{k'}-1=0$.

The general equation of conics cutting the axes of coordinates where these lines meet them is

$$\left(\frac{x}{h}+\frac{y}{k}-1\right)\left(\frac{x}{h'}+\frac{y}{k'}-1\right)=Axy,$$

which is of the form

$$\frac{x^2}{hh'}+\lambda xy+\frac{y^2}{kk'}-x\left(\frac{1}{h}+\frac{1}{h'}\right)-y\left(\frac{1}{k}+\frac{1}{k'}\right)+1=0.$$

Examples. 1. The polar of a given point with respect to a system of conics passing through four given points will pass through a fixed point.

2. The locus of the centres of conics through four given points is a conic whose asymptotes are parallel to the axes of the two parabolas through the four given points.

EXAMPLES.

1. Shew that the foci of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are given by

$$Fx + Gy - H = Cxy,$$
$$2Gx - 2Fy - A + B = C(x^2 - y^2),$$

the axes being rectangular.

[We have seen that the tangents from a focus (x_1, y_1) are

$$[(x - x_1) + i(y - y_1)][(x - x_1) - i(y - y_1)] = 0.$$

Now the condition that the line $lx + my + n = 0$ should be a tangent to the conic is (§ 133)

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

Thus the condition that $x + iy - (x_1 + iy_1) = 0$ should be a tangent is got by writing $l = 1$, $m = i$ and $n = -(x_1 + iy_1)$, that is

$$A - B + C(x_1 + iy_1)^2 - 2iF(x_1 + iy_1) - 2G(x_1 + iy_1) + 2iH = 0.$$

That is

$$A - B + C(x_1^2 - y_1^2) + 2Fy_1 - 2Gx_1 + 2i\{Cx_1y_1 - Fx_1 - Gy_1 + H\} = 0.$$

Similarly the condition that $x - iy - (x_1 - iy_1) = 0$ should be a tangent is

$$A - B + C(x_1^2 - y_1^2) + 2Fy_1 - 2Gx_1 - 2i\{Cx_1y_1 - Fx_1 - Gy_1 + H\} = 0.$$

Adding and subtracting these, we see that the foci satisfy the equations

$$A - B - 2Gx + 2Fy + C(x^2 - y^2) = 0,$$
$$H - Fx - Gy + Cxy = 0.]$$

2. The equation of the director circle of the conic given by the general equation $S = 0$ when the axes of coordinates are inclined at an angle ω is

$$(a + b - 2h\cos\omega)S = \xi^2 + \eta^2 - 2\xi\eta\cos\omega$$

where $\qquad \xi = ax + hy + g \quad$ and $\quad \eta = hx + by + f,$

and the equation of a pair of directrices is of the form

$$kS + \xi^2 + \eta^2 - 2\xi\eta\cos\omega = 0.$$

3. Four points A, B, C, D on an ellipse are concyclic. The circle which passes through A and touches the curve in B cuts it again in B'. Similarly circles through A which touch the ellipse in C and D cut it in C' and D'. Shew that A, B', C', D' are concyclic.

4. Find the equation of the parabola which touches the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

at the points where it is cut by the line $lx + my + n = 0$.

5. A family of conics have double contact with a given conic at the extremities of a given chord. Prove that the locus of the centres of conics of the family is the diameter of the given conic conjugate to the given chord.

6. Prove that the locus of a point, the sum or difference of the tangents from which to two given circles is constant is a conic having double contact with each of the two circles.

7. At the point $(a\cos\phi, b\sin\phi)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the parabola of four point contact is drawn. Shew the equation of its axis is

$$\frac{x}{a}\sin\phi - \frac{y}{b}\cos\phi = \frac{(a^2 - b^2)\sin\phi\cos\phi}{a^2\cos^2\phi + b^2\sin^2\phi}.$$

8. Prove that if two conics have double contact the polar of the centre of either with respect to the other is parallel to the chord of contact.

9. Two tangents are drawn from a variable point P to a conic to meet the tangent at a fixed point Q in T and T'. Shew that if $QT \cdot QT'$ is constant the locus of P is a straight line parallel to the tangent at Q.

10. A circle touches a hyperbola at two points, the chord of contact being parallel to the transverse axis. Prove that the ratio of the length of the tangent to the circle from any point of the conic to the distance of the point from the chord of contact is the eccentricity of the conjugate hyperbola.

11. Tangents are drawn from a point O to an ellipse so as to intercept a fixed length on the tangent at a fixed point P; prove that the locus of O is a conic which has four point contact with the ellipse at the other extremity P' of the diameter through P.

[Take as coordinate axes the tangent at P and the diameter through P.]

12. Prove that the locus of the centres of non-degenerate conics having four point contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\alpha, b\sin\alpha)$ is the line $\frac{x\sin\alpha}{a}-\frac{y\cos\alpha}{b}=0$.

13. If a system of conics be drawn having four point contact with the conic $ax^2+2hxy+by^2+2fy=0$ at the origin, prove that the director circles of these conics form a coaxial system whose limiting points are the origin and the point $\left(-\frac{fh}{a^2+h^2}, \frac{af}{a^2+h^2}\right)$.

14. A conic is drawn to have double contact with each of two given circles. Shew that the two chords of contact are either parallel to each other and to the radical axis and equidistant from it; or are perpendicular to each other and intersect in a limiting point of the two circles.

15. The foot N of the ordinate of a point P on a parabola is the centre of the circle of curvature at its vertex. Prove that the centre of curvature at P lies on the parabola.

16. Four common tangents are drawn to an ellipse and a parabola having its focus at the centre of the ellipse. Prove that the sum of the eccentric angles of the points of contact of the tangents with the ellipse is an even multiple of π.

17. Two chords AB, CD of a rectangular hyperbola make angles α and β with the transverse axis. Prove that the angle between the axes of the parabolas through A, B, C, D is

$$\cos^{-1}\{\sin(\alpha+\beta)\sec(\alpha-\beta)\}.$$

18. From any point P of an ellipse perpendiculars PM, PN are let fall on the axes and MN meets the ellipse in Q and Q'; prove that the rectangular hyperbola which touches the ellipse at P and meets it in Q and Q' has its asymptotes parallel to the axes of the ellipse and that its curvature at P is double that of the ellipse.

19. If a parabola touch a fixed circle and pass through the extremities of a fixed diameter; prove that its axis will pass through one or other of two fixed points.

20. Three points (x_1, y_1), (x_2, y_2), (x_3, y_3) on the parabola $y^2 = 4ax$ are such that their centres of curvature are collinear; prove that $\dfrac{1}{y_1} + \dfrac{1}{y_2} + \dfrac{1}{y_3} = 0$.

21. Prove that the product of the radii of curvature at the feet of the normals drawn from any point to a parabola is eight times the cube of the distance of that point from the focus.

22. Prove that if the circles of curvature at the extremities of the major axis of an ellipse pass through the centre, then no other circles of curvature at real points of the ellipse pass through the centre.

23. Normals are drawn from any point (ξ, η) to the parabola $y^2 = 4ax$. Prove that ρ_1, ρ_2, ρ_3 the radii of curvature at the feet of the normals satisfy

$$\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} + \rho_3^{\frac{2}{3}} = 4^{\frac{1}{3}} a^{-\frac{1}{3}} (2\xi - a).$$

24. Parabolas are drawn with their vertices at the origin and their axes along the axis of x; tangents are drawn to them from a fixed point $(f, 0)$. Shew that the locus of their centres of curvature at the points of contact is the curve

$$(x + 3f)\, y^2 + 8f^3 = 0.$$

25. Prove that the locus of the pole of the axis of x with respect to the circle of curvature at any point of the parabola $y^2 = 4ax$ is

$$(x - 2a)^3 y^2 = 12a\,(x^2 - ax + a^2)^2.$$

26. Three points on an ellipse (semi-axes a and b) are situated so that the circles of curvature at those points all cut the ellipse again at the same point. If their radii are ρ_1, ρ_2, ρ_3, then

$$\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} + \rho_3^{\frac{2}{3}} = \frac{3\,(a^2 + b^2)}{2a^{\frac{2}{3}}\, b^{\frac{2}{3}}}.$$

27. Prove that there are four points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (other than the point P) such that the centre of curvature at each of them lies on the normal at P.

Further prove that the normals at these four points meet in a point on the ellipse

$$a^2x^2 + b^2y^2 = (a^2 - b^2)^2.$$

28. Prove that the locus of the centres of the rectangular hyperbolas which have contact of the third order with a given parabola is an equal parabola having the same axis and directrix.

29. The equation of the latus rectum of the parabola

$$(ax + \beta y)^2 + 2gx + 2fy + c = 0$$

is
$$2(\beta g - af)(\beta x - ay) + (a^2 + \beta^2)c + f^2 + g^2 - 2\frac{(ag + \beta f)^2}{a^2 + \beta^2} = 0.$$

30. Two families of parallel straight lines are given and also a conic in the same plane. Prove that the locus of points such that the lines through them are conjugate with respect to the conic is a hyperbola whose asymptotes are the lines through the centre of the conic.

31. Shew that if a parabola be referred to a tangent and normal as axes of x and y, its equation may be put in the form

$$(\beta x + ay)^2 = 4\beta y(a^2 + \beta^2),$$

and the equation of its directrix in the form $ax - \beta y = a^2 + \beta^2$ where a and β are the coordinates of its focus.

32. An ellipse whose semi-axes are a and b touches the axis of x at the origin; prove that the locus of its centre is

$$x^2y^2 + (y^2 - a^2)(y^2 - b^2) = 0.$$

33. Lines OP, OQ at right angles are drawn through a fixed point O to meet a conic in P and Q. Shew that the locus of the foot of the perpendicular from O on the chord PQ is a circle.

34. Three lines OPP', OQQ', ORR' through a point O meet a conic in P and P', Q and Q', R and R' respectively. Prove that with a suitable convention as to signs

$$\left(\frac{1}{OP} + \frac{1}{OP'}\right)\sin Q\hat{O}R + \left(\frac{1}{OQ} + \frac{1}{OQ'}\right)\sin R\hat{O}P + \left(\frac{1}{OR} + \frac{1}{OR'}\right)\sin P\hat{O}Q = 0.$$

35. POP' is a chord of a conic which is bisected in O. If QOQ', ROR' are chords making equal angles with POP', shew that $\frac{1}{OQ} \sim \frac{1}{OQ'} = \frac{1}{OR'} \sim \frac{1}{OR}$.

36. A family of conics have double contact with a given conic at the extremities of any chord. Prove that the locus of the centres of conics of the family is the diameter of the given conic conjugate to the chord.

37. Prove that if the bisectors of the internal and external angles between two tangents to a conic be parallel to two given diameters of the conic, the point of intersection of the tangents lies on a conic.

38. Shew that if $y = tx$ is a normal to the general conic given by the general equation, then t is a root of the equation

$$t^4(gG + hH) + 2t^3(aH + hA) + t^2\{(a - b)(A - B) \\ + fF + gG - 2hH\} + 2t(bH + hB) + fF + hH = 0.$$

39. The focus of the parabola $\sqrt{\frac{x}{h}} + \sqrt{\frac{y}{k}} = 1$ referred to two tangents inclined at an angle ω is given by the equations

$$hx = ky = x^2 + 2xy\cos\omega + y^2.$$

40. The directrix of the parabola $\sqrt{ax} + \sqrt{by} = 1$ is

$$(b + a\cos\omega)x + (a + b\cos\omega)y = \cos\omega$$

and the length of the latus rectum is

$$\frac{4ab\sin^2\omega}{(a^2 + b^2 + 2ab\cos\omega)^{\frac{3}{2}}}.$$

41. An ellipse of semi-axes a and b slides between two lines inclined to each other at an angle 2α; shew that the locus of its centre referred to the bisectors of the angles between the lines as axes is

$$(x^2\cos^4\alpha + y^2\sin^4\alpha)(x^2 + y^2) - (a^2 + b^2)(x^2\cos^2\alpha + y^2\sin^2\alpha) \\ + (a^2\cos^2\alpha + b^2\sin^2\alpha)(a^2\sin^2\alpha + b^2\cos^2\alpha) = 0.$$

42. The equation of the equiconjugate diameters of the conic whose equation referred to rectangular axes is

$$ax^2 + 2hxy + by^2 = 1$$

is $\quad (a^2 - ab + 2h^2)x^2 + 2h(a + b)xy + (b^2 - ab + 2h^2)y^2 = 0.$

43. Prove that if the conics $S=0$, $S'=0$ have a pair of common chords $\alpha=0$, $\beta=0$ such that $S-S'\equiv\alpha\beta$, the equation

$$k^2\alpha^2-2k(S+S')+\beta^2=0$$

represents a conic having double contact with each of the conics S and S'.

A conic has finite double contact with each of the conics

$$x^2+y^2-e^2(x+c)^2=0, \qquad x^2+y^2-e'^2(x+c)^2=0.$$

Write down its general equation and prove that the chords of contact are perpendicular chords through the origin; also that if $e^{-2}+e'^{-2}=1$ all such conics are rectangular hyperbolas.

44. At the point P (eccentric angle ϕ) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ the parabola having contact of the third order is drawn; shew that the equation of its directrix is

$$2ax\cos\phi+2by\sin\phi=a^2(1+\cos^2\phi)+b^2(1+\sin^2\phi).$$

45. A series of circles with a given centre O are drawn. Shew that the middle point of their chords of intersection with a given conic lie on a rectangular hyperbola whose asymptotes are parallel to the axes of the conic and which passes through O and the centre of the conic.

46. Points P, Q, R are taken on an ellipse in such a manner that the tangent at each point is parallel to the line joining the other two. Shew that the locus of the centre of mean position of the centres of curvature at P, Q, R is

$$16(a^2x^2+b^2y^2)=(a^2-b^2)^2.$$

47. Shew that through any point (f, g) in the plane of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ there pass six of the circles of curvature, and verify that the six centres of curvature lie on the conic

$$\{2(f^2+g^2-2fx-2gy)-a^2-b^2\}^2=12(a^2x^2+b^2y^2)-3(a^2-b^2)^2.$$

48. Shew that the conic which is coaxial with

$$ax^2+2hxy+by^2+2gx+2fy=0$$

and passes through the foci is

$$(a-b)\left[C(x^2-y^2)-2Gx+2Fy+A-B\right]+4h\left[Cxy-Fx-Gy+H\right]=0.$$

49. Find the locus of the foot of the perpendicular drawn from a point situated on a given straight line, to its polar with regard to a central conic; and shew that it passes through the foci of the conic.

50. Shew that the equations of the two pairs of conjugate diameters of the conic given by the general equation, which include an angle θ between them, are

$$(bX^2 - 2hXY + aY^2)\cos\theta + \{h(X^2 - Y^2) - (a - b)XY\}\sin\theta = 0,$$

where $$X = ax + hy + g, \quad Y = hx + by + f,$$

the axes of coordinates being rectangular.

CHAPTER XIII.

SIMILAR CONICS AND CONFOCAL SYSTEMS.

240. Definition.

If S be a plane figure and P any point of it, and O a fixed point in its plane, and if on OP a point P' be taken such that $OP' : OP = k$ a positive constant, the locus of the point P' thus determined will be a new figure S' which is said to be *similar and similarly situated* to S; two such figures S and S' are conveniently called *homothetic* figures and the point O is the *homothetic centre.*

Suppose now that in the above construction for the figure S' from the figure S, k instead of being a *positive* constant is a negative constant, the figure S' which is the locus of P' is then said to be *antihomothetic* to S.

Next suppose that k is imaginary, then the figure S' may be said to be *imaginarily homothetic* with S.

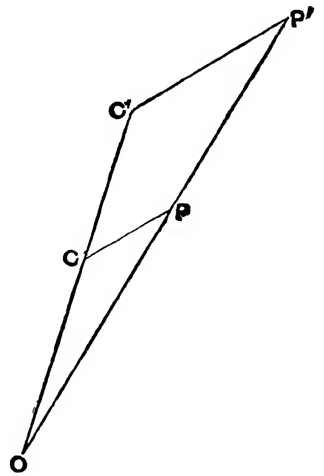
241. Proposition. *If S and S' be two coplanar homothetic figures, C and C' fixed corresponding points of them, P and P' any other pair of corresponding points, not in the line of C and C', then CP and $C'P'$ will be parallel and the ratio $C'P' : CP$ will be constant $(=k)$ for the various positions of P and P'.*

For let O be the homothetic centre. Then

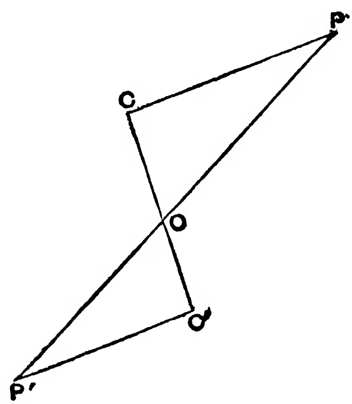
$$OC' : OC = k = OP' : OP,$$

$\therefore$ $C'P'$ and CP are parallel, and $C'P' : CP = OP' : OP = k.$

COR. *This proposition holds equally well if S and S' be imaginarily homothetic, or if they be antihomothetic.*



In this latter case O will lie between C and C' as in figure.



242. Proposition. *If S be a plane figure, C and C' two points in its plane and if C be joined to any point P on S and $C'P'$ be drawn parallel to CP and such that $C'P' : CP = k$, a positive constant, then the figure S' which is the locus of P' will be homothetic with S.*

For (see figure of previous article) let CC' and PP' meet in O, then since CP and $C'P'$ are parallel we have

$$OC' : OC = C'P' : CP = k.$$

Thus O is a fixed point.

Also $$OP' : OP = C'P' : CP = k.$$

Therefore the locus of P' is a figure homothetic with S, the locus of P, and O is the homothetic centre.

COR. *If k be a negative constant the figure S' is antihomothetic with S, and if k be imaginary S' is imaginarily homothetic with S.*

243. Proposition. *If S be a central conic and S' be constructed really homothetic with S, or antihomothetic, or imaginarily homothetic with it, then S' will be a conic having its asymptotes parallel to those of S.*

For let the equation of S referred to Cartesian axes at O, the homothetic centre, be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1).}$$

Let (X, Y) be the coordinates of the point P' on S' corresponding to $P\ (x, y)$ on S.

Then clearly $$X = kx, \quad Y = ky,$$

where k is the constant ratio $OP' : OP$.

Therefore

$$aX^2 + 2hXY + bY^2 + k(2gX + 2fY) + k^2c = 0.$$

That is to say, the locus of P' is the conic

$$ax^2 + 2hxy + by^2 + k(2gx + 2fy) + k^2c = 0 \text{.........(2).}$$

Now both (1) and (2) have their asymptotes parallel to the lines

$$ax^2 + 2hxy + by^2 = 0.$$

Thus S' is a conic having its asymptotes parallel to those of S.

COR. 1. *The axes of S' will be parallel to those of S.*

For the axes bisect the angles between the asymptotes. But it must not be assumed that *corresponding* axes of the two conics are parallel.

COR. 2. *The centres of the two conics S and S' are corresponding points.*

For the centre of (1) is given by

$$\left.\begin{array}{l} ax + hy + g = 0 \\ hx + by + f = 0 \end{array}\right\},$$

and the centre of (2) is given by

$$\left.\begin{array}{l} ax + hy + kg = 0 \\ hx + by + kf = 0 \end{array}\right\}.$$

From these we see that if (x_s, y_s) $(x_{s'}, y_{s'})$ be the centres of S and S' respectively,

$$x_{s'} = kx_s \quad \text{and} \quad y_{s'} = ky_s.$$

Therefore the centres are corresponding points.

COR. 3. *If S, instead of being a central conic, be a parabola, then S' will be a parabola having its axis parallel to that of S.*

For the equation of S will now be of the form

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0,$$

while that of S' will be

$$(\alpha x + \beta y)^2 + k(2gx + 2fy) + k^2 c = 0.$$

This is also a parabola, and both S and S' have their axes parallel to the line

$$\alpha x + \beta y = 0.$$

Examples. 1. Shew that the pair of tangents from O, the homothetic centre to each of the two conics S and S', which are really homothetic, antihomothetic or imaginarily homothetic are the same lines.

2. Prove that the tangents to the homothetic conics S and S' at corresponding points of them are parallel.

244. We have seen that if two central conics are homothetic they have their asymptotes parallel. We shall now go on to shew that *if two central conics have their asymptotes parallel they are homothetic.*

For let the equations of the two conics S and S' *referred to any axes be*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)},$$

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \quad \text{.........(2)}.$$

As the asymptotes are parallel we have

$$\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'} = \lambda \text{ (say)}.$$

The axes of (1) are parallel to the axes of (2). Transforming (1) to principal axes, the equation of S becomes (§ 220)

$$\alpha x^2 + \beta y^2 + \frac{\Delta}{ab - h^2} = 0 \quad \text{.................(3)}.$$

Now the equation (2) can be written

$$ax^2 + 2hxy + by^2 + 2\lambda g'x + 2\lambda f'y + \lambda c' = 0,$$

which, when we refer the conic to its principal axes, will become

$$\alpha x^2 + \beta y^2 + \frac{L}{ab - h^2} = 0,$$

where

$$L = \begin{vmatrix} a, & h, & \lambda g' \\ h, & b, & \lambda f' \\ \lambda' g', & \lambda f', & \lambda c' \end{vmatrix} = \begin{vmatrix} \lambda a', & \lambda h', & \lambda g' \\ \lambda h', & \lambda b_1, & \lambda f' \\ \lambda g', & \lambda f', & \lambda c' \end{vmatrix} = \lambda^3 \Delta',$$

that is we have as the equation of S'

$$\alpha x^2 + \beta y^2 + \frac{\lambda^3 \Delta'}{ab - h^2} = 0 \quad \text{...............(4)}.$$

The axes of coordinates for equation (4) are parallel to those for equation (3) but not coincident with them.

Take two parallel radii vectores making an angle θ with the x axis; denoting them by r and r' we have

$$(\alpha \cos^2 \theta + \beta \sin^2 \theta) r^2 + \frac{\Delta}{ab - h^2} = 0,$$

$$(\alpha \cos^2 \theta + \beta \sin^2 \theta) r'^2 + \frac{\lambda^3 \Delta'}{ab - h^2} = 0,$$

$$\therefore r^2 : r'^2 = \Delta : \lambda^3 \Delta'.$$

Thus the two conics will be homothetic or antihomothetic, and if the former they will be *really* homothetic if $\Delta : \lambda^2\Delta'$ be positive, that is if $\Delta : \Delta'$ have the same sign as λ. They will be *imaginarily* homothetic if the ratio $\Delta : \Delta'$ have the opposite sign to that of λ.

We shall shew presently that if two central conics are antihomothetic they are homothetic also. Assuming this for the present, we see that two central conics with their asymptotes parallel are homothetic; whether they are really or imaginarily homothetic depends on the sameness or difference of sign of the ratio $\Delta : \Delta'$ and the ratio $a : a'$.

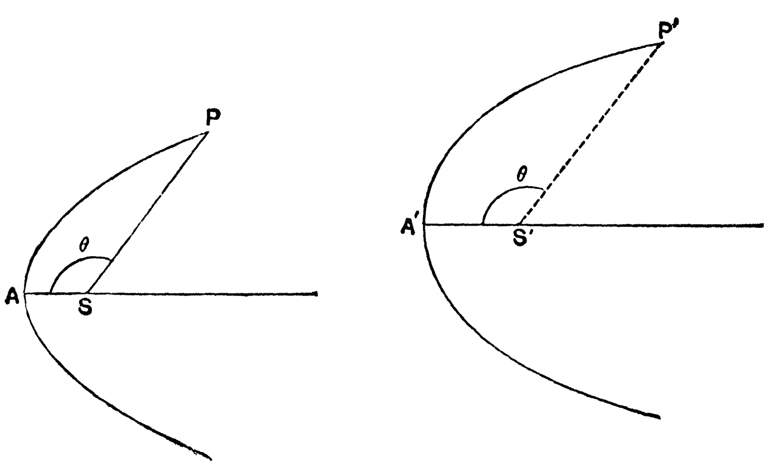
245. Proposition. *Two parabolas will be really homothetic or antihomothetic if they have their axes parallel.*

First let the curvatures at the vertices A and A' be in the same directions. Take SP and $S'P'$, two focal radii vectores making the same angle θ with SA and $S'A'$ respectively, the lines joining the foci S and S' to the vertices.

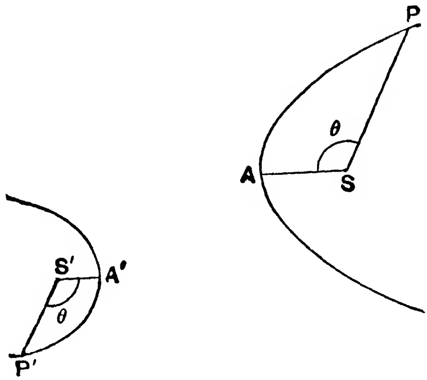
Then if l and l' be the semi-latera recta,

$$\frac{l}{SP} = 1 + \cos\theta = \frac{l'}{S'P'},$$

$$\therefore SP : S'P' = l : l'.$$



Thus the parabolas are homothetic, and clearly the foci and vertices are corresponding points in the two figures.



Next let the curvatures at A and A' be in opposite directions.

Let SP and $S'P'$ be two parallel focal radii vectores making the same angle θ with SA and $S'A'$ but measured in opposite directions.

Then $$\frac{l}{SP} = 1 + \cos\theta = \frac{l}{S'P'},$$

$$\therefore SP : S'P' = l : l'.$$

Thus as SP and $S'P'$ are in opposite directions the two parabolas are antihomothetic.

Figures directly similar but not similarly situated.

246. Def. Two coplanar figures S and S' are said to be directly similar when one of them (S' say) can be rotated about a point (A' say) in the plane so as to become homothetic with the other (S).

Let P' and Q' be two points of S'. Let these points when S' is rotated round A' through the proper angle (α say) come into the positions P'', Q'' so as to be homothetic with P and Q of the figure S.

Then $$P''Q'' : PQ = k.$$

But since $\quad A'P' = A'P''$ and $A'Q' = A'Q''$,

and $\quad \angle P'A'Q' = \angle P''A'Q''$ for $\angle P'A'P'' = \angle Q'A'Q''$,

$$\therefore \triangle P'A'Q' \equiv \triangle P''A'Q'',$$

and $$P'Q' = P''Q'',$$

thus $$P'Q' : PQ = k.$$

Thus the line joining any two points of the figure S is in a constant ratio to the line joining the two corresponding points of S'. And therefore the triangle formed by joining any three points of S is similar to the triangle formed by joining the three corresponding points of S'.

247. Proposition. *If S and S' be two coplanar figures which are directly similar and α be the angle through which S' must be turned round the point A' to make it homothetic with S, then if S' be turned about any other point B' in the plane through the same angle α it will become homothetic with S.*

For let S_1 be the new figure when this rotation round B' takes place and let the point P' of S' become P_1 of S_1.

Now plainly if we rotate S' about any point through an angle α the line $P'Q'$ joining two points of S' will come into a position which will be parallel to the line joining the points P and Q of S which correspond with P' and Q'. Therefore P_1Q_1 is parallel to PQ.

And $$P_1Q_1 = P'Q' = k \,.\, PQ.$$

Thus the ratio of $P_1Q_1 : PQ$ is constant.

Hence the figure S_1 may be regarded as the locus of points Q_1 got by drawing lines P_1Q_1 through P_1 parallel to PQ for various positions of Q in the figure S, and such that $P_1Q_1 = PQ$ is constant $(= k)$.

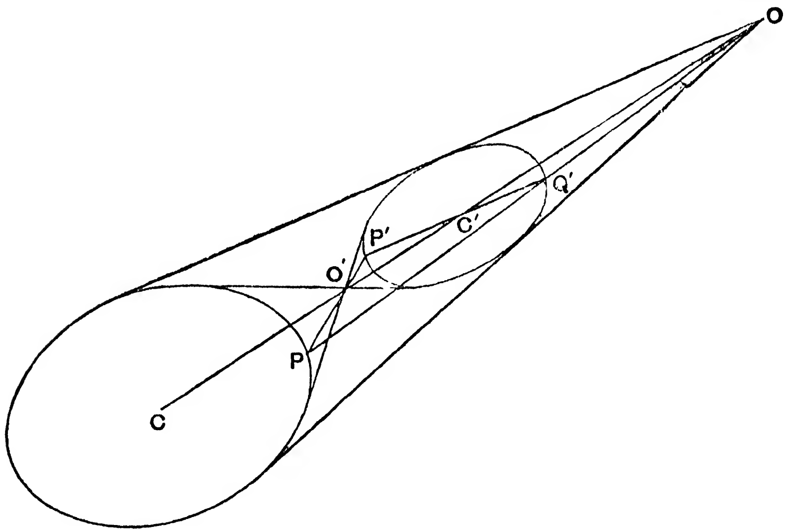
Thus S_1 is homothetic with S.

248. Proposition. *If two central conics S and S' are antihomothetic they are at the same time also homothetic.*

For two antihomothetic figures are such that if one of them be rotated through two right angles it will become homothetic with the other.

Hence if two central conics S and S', be antihomothetic, then S' by rotation round its centre through two right angles will become homothetic with S'. But when a central conic is rotated about its centre through two right angles the opposite ends of each diameter merely exchange places.

So that if $P'Q'$ be a diameter of S', and P be the point of S corresponding with P' when the conics are considered as antihomothetic, then S and S' must also be homothetic, with P and Q' as corresponding points.



249. *To find the condition that the conics*

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$
$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

may be similar but not homothetic.

If the conics are similar then S' by rotation about any point in its plane can be made homothetic with S, and then the asymptotes of the two will be parallel.

Hence in their original positions the angle between the asymptotes of the one must be equal to the angle between the asymptotes of the other.

But the asymptotes of S and S' are parallel respectively to

$$ax^2 + 2hxy + by^2 = 0,$$
$$a'x^2 + 2h'xy + b'y^2 = 0.$$

Hence we must have (§ 110)

$$\frac{2\sqrt{ab - h^2}\sin\omega}{a + b - 2h\cos\omega} = \pm\frac{2\sqrt{a'b' - h'^2}\sin\omega}{a' + b' - 2h'\cos\omega},$$

where ω is the angle between the axes.

$$\frac{ab - h^2}{(a + b - 2h\cos\omega)^2} = \frac{a'b' - h'^2}{(a' + b' - 2h'\cos\omega)^2}.$$

This then is the *necessary* condition that the conics should be similar. It is also *sufficient* for, if it hold, the angle between the asymptotes of the one will be equal to the angle between the asymptotes of the other.

And thus by rotation of one of the conics it can be brought into a position such that its asymptotes will be parallel to those of the other, that is the two will be homothetic.

The condition however does not discriminate between the two cases where S and S' are *really* similar and where they are only imaginarily similar.

Examples. 1. Two conics whose eccentricities are equal are similar.

2. A hyperbola and its conjugate are imaginarily homothetic, and a real focus of the one corresponds with an imaginary focus of the other.

250. Confocal conics.

A system of central conics having all four foci in common is called a confocal system.

The general equation of conics confocal with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is at once seen to be (§ 215)

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1.$$

Different confocals are obtained by taking different values of λ.

251. Proposition. *Through every point in the plane of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *two confocal conics can be drawn, one an ellipse and the other a hyperbola.*

For the value of λ for a confocal through the point (x_1, y_1) is given by

$$\frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1.$$

That is

$$(\lambda+a^2)(\lambda+b^2)-x_1^2(\lambda+b^2)-y_1(\lambda+a^2)=0 \quad \ldots(1).$$

Now when $\lambda=\infty$ the left-hand side of this is positive,

when $\lambda=-b^2$ the left-hand side is negative,

when $\lambda=-a^2$ the left-hand side is positive.

Thus there is a value of λ between ∞ and $-b^2$ satisfying (1) and a value between $-b^2$ and $-a^2$.

That is one of the values of λ satisfying (1) makes both $a^2+\lambda$ and $b^2+\lambda$ positive, and the other makes $a^2+\lambda$ positive and $b^2+\lambda$ negative.

Hence there are two confocals through (x_1, y_1) to the given ellipse, the one an ellipse and the other a hyperbola.

252. Proposition. *Confocal conics cut at right angles.*

Let
$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

be the two conics through the point (x_1, y_1) confocal with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Therefore
$$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1$$

and
$$\frac{x_1^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

whence by subtraction

$$\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}=0,$$

which is the condition that the tangents at (x_1, y_1) to (1) and (2), viz.,

$$\frac{xx_1}{a^2+\lambda_1}+\frac{yy_1}{b^2+\lambda_1}=1$$

and
$$\frac{xx_1}{a^2+\lambda_2}+\frac{yy_1}{b^2+\lambda_2}=1$$

should be at right angles to one another.

Hence the proposition is proved.

253. *To express the coordinates of any point in the plane of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *in terms of the axes of the conics through it confocal with the ellipse.*

Let (x_1, y_1) be the point and let the confocals through it be

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1,$$

$$\frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1,$$

$$\therefore\ \frac{x_1^2}{a^2+\omega}+\frac{y_1^2}{b^2+\omega}-1$$

vanishes when $\omega=\lambda_1$ and when $\omega=\lambda_2$,

$$\therefore\ \frac{x_1^2}{a^2+\omega}+\frac{y_1^2}{b^2+\omega}-1\equiv\frac{A(\lambda_1-\omega)(\lambda_2-\omega)}{(a^2+\omega)(b^2+\omega)},$$

where A is independent of ω.

Put $\omega=\infty$ and we get $A=-1$,

$$\therefore\ \frac{x_1^2}{a^2+\omega}+\frac{y_1^2}{b^2+\omega}-1\equiv-\frac{(\lambda_1-\omega)(\lambda_2-\omega)}{(a^2+\omega)(b^2+\omega)},$$

$$\therefore\ x_1^2+\frac{y_1^2(a^2+\omega)}{b^2+\omega}-(a^2+\omega)\equiv-\frac{(\lambda_1-\omega)(\lambda_2-\omega)}{b^2+\omega}.$$

In this put $\omega=-a^2$,

$$\therefore\ x_1^2=\frac{(a^2+\lambda_1)(a^2+\lambda_2)}{a^2-b^2}=\frac{a_1^2a_2^2}{a^2-b^2}.$$

Similarly $\quad y_1^2 = \dfrac{(b^2+\lambda_1)(b^2+\lambda_2)}{(b^2-a^2)} = \dfrac{b_1^2 b_2^2}{b^2-a^2},$

where
$$a_1^2 \equiv a^2+\lambda, \quad a_2^2 = a^2+\lambda_2,$$
$$b_1^2 = b^2+\lambda_1, \quad b_2^2 = b^2+\lambda_2.$$

The same result could have been obtained by solving the equations

$$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1,$$
$$\frac{x_1^2}{a^2+\lambda_2}+\frac{y_1^2}{b^2+\lambda_2}=1.$$

254. Let p_1 and p_2 be the perpendiculars from the centre on the tangents at P (x_1, y_1) to the two confocals through (x_1, y_1),

$$\therefore \frac{1}{p_1^2} = \frac{x_1^2}{(a^2+\lambda_1)^2}+\frac{y_1^2}{(b^2+\lambda_1)^2}$$
$$= \frac{\dfrac{a_1^2 a_2^2}{a^2-b^2}}{a_1^4} - \frac{\dfrac{b_1^2 b_2^2}{a^2-b^2}}{b_1^4}$$
$$= \frac{1}{a^2-b^2}\cdot\left(\frac{a_2^2}{a_1^2}-\frac{b_2^2}{b_1^2}\right)$$
$$= \frac{1}{a^2-b^2}\cdot\frac{(a^2+\lambda_2)(b^2+\lambda_1)-(b^2+\lambda_2)(a^2+\lambda_1)}{a_1^2 b_1^2}$$
$$= \frac{\lambda_1-\lambda_2}{a_1^2 b_1^2} = \frac{a_1^2-a_2^2}{a_1^2 b_1^2},$$
$$\therefore p_1^2 = \frac{a_1^2 b_1^2}{a_1^2-a_2^2}.$$

Similarly $\quad p_2^2 = \dfrac{a_2^2 b_2^2}{a_2^2-a_1^2}.$

Example. If P be a point in the plane of a conic S, centre C, and if a_1, b_1 and a_2, b_2 be the semi-axes of the two confocals to S through P and if conics be drawn having their centre at P and their semi-axes of lengths a_1, a_2 and b_1, b_2 lying along the normals to the two confocals through P, these two conics thus drawn will be confocal with one another and will pass through C and have for their tangents at C the axes of S. [The coordinates of C with reference to the axes of the new coordinates will be p_1 and p_2 (§ 254).]

255. Confocal parabolas. Two parabolas may be regarded as confocal if they have a common focus and their axes in the same line. Two confocal parabolas may be looked upon as the limiting case of two confocal ellipses, three of whose foci are at infinity.

A system of confocal parabolas is best represented by taking their common non-vanishing focus for the origin and their common axis for the axis of x. Then if $4a$ be the latus rectum of one of the parabolas its equation will be

$$y^2 = 4a(x + a).$$

This equation then gives for different values of a a system of confocal parabolas.

EXAMPLES.

1. If the confocals through (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are
$$\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1 \text{ and } \frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1,$$

then
$$\text{(i)}\quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = -\frac{\lambda_1\lambda_2}{a^2b^2},$$

$$\text{(ii)}\quad x_1^2 + y_1^2 - a^2 - b^2 = \lambda_1 + \lambda_2.$$

2. If ϕ be the angle between the tangents from (x_1, y_1) to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

prove that
$$\tan\phi = \frac{2ab\sqrt{\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1}}{x_1^2 + y_1^2 - a^2 - b^2}.$$

Hence shew that

$$\tan\frac{\phi}{2} = \sqrt{-\frac{\lambda_2}{\lambda_1}},$$

where λ_1 and λ_2 are the parameters of the confocals through P.

3. The equation of the pair of tangents from P to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

referred to the normals to the confocals through P as axes is

$$\frac{X^2}{\lambda_1}+\frac{Y^2}{\lambda_2}=0$$

where λ_1 and λ_2 are the parameters of the confocals through P.

4. Shew that if ψ be the angle which the tangents from P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make with the tangent at P to the confocal ellipse through P, then

$$\sin\psi=\sqrt{\frac{\lambda_1}{\lambda_1-\lambda_2}},$$

where λ_1 and λ_2 are the parameters of the confocal ellipse and hyperbola respectively.

5. If from a point P tangents be drawn to each of two confocal ellipses and ψ and ψ' be the angles which these make with the tangent at P to the confocal ellipse through P, then the ratio $\sin\psi:\sin\psi'$ will be constant for all positions of P on an ellipse confocal with the given ellipses.

6. If two similar concentric ellipses touch one another, shew that the angle between their major axes is

$$\tan^{-1}\frac{b\,(a'^2-a^2)}{a\sqrt{(a'^2-b^2)\,(a^2-b'^2)}},$$

a, b and a', b' being the semi-axes.

7. The locus of the pole of a given straight line with respect to a system of confocal conics is a straight line.

8. The difference of the squares of the perpendiculars drawn from the centre on any two parallel tangents to two given confocals is constant.

[Take $x\cos\alpha+y\sin\alpha=p$ and $x\cos\alpha+y\sin\alpha=p'$ as the tangents to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\text{and}\quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.]$$

9. The locus of points such that two tangents drawn from them, one to each of two confocals, are at right angles is a circle concentric with the two confocals.

10. Tangents are drawn through a fixed point (x', y') to a series of conics confocal with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: shew that the locus of their points of contact is

$$\frac{x}{y-y'} + \frac{y}{x-x'} = \frac{a^2 - b^2}{x'y - xy'}.$$

11. Shew that the locus of a point such that the tangents from it to the ellipse $x^2/a^2 + y^2/b^2 = 1$ contain an angle 2α is given by

$$a_1^2 \cos^2 \alpha + a_2^2 \sin^2 \alpha = a^2,$$

where a_1 and a_2 are the primary semi-axes of the confocals through the point.

12. Shew that the tangents at (h, k) to the conics passing through (h, k) confocal with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are represented by the equation

$$hk\{(x-h)^2 - (y-k)^2\} + (k^2 - h^2 + a^2 - b^2)(x-h)(y-k) = 0.$$

13. From a fixed point P pairs of tangents PQ, PQ' are drawn to each of a system of confocal conics. Prove that the circles PQQ' form a coaxal system.

14. A straight line $PQRS$ intersects two confocal ellipses at the points P, Q, R, S in order. If the tangents at P and R are perpendicular, prove that the tangents at Q and S are also perpendicular.

15. The polar of a fixed point (ξ, η) with respect to one conic of a system confocal with $x^2/a^2 + y^2/b^2 = 1$ touches another conic of the system at some point. Shew that the locus of such points is the curve

$$(\xi y - \eta x)(x^2 + y^2 - \xi x - \eta y) = (a^2 - b^2)(x - \xi)(y - \eta).$$

16. Prove that the two families of conics which have their centres at the point (α, β) and touch the axes of x and y respectively at the origin are represented by the equations

$$(\alpha y - \beta x)^2 = A(y^2 - 2\beta y)$$

and

$$(\alpha y - \beta x)^2 = B(x^2 - 2\alpha x)$$

respectively, A and B having arbitrary values. Prove also that if $A + B = 0$ the pairs of conics are confocal.

17. From a point P on an ellipse of semi-axes a' and b' tangents are drawn to an internal confocal ellipse, and these meet the first ellipse in points Q and R; prove that the locus of the intersections of tangents at Q and R is another ellipse of semi-axes a and b such that

$$\frac{1}{a'^2}\left(1-\frac{b}{b'}\right)=\frac{1}{b'^2}\left(1-\frac{a}{a'}\right).$$

18. Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from any point T on a given hyperbola confocal with the ellipse; if 2θ be the angle between the tangents, prove that $\sin\theta$ varies inversely as CD, where CD is the semi-diameter conjugate to CT of the ellipse through T confocal with the given one.

19. If two conics be drawn confocal with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

respectively, then the eight points of contact of their common tangents will lie on a circle.

20. If a triangle be inscribed in a conic and circumscribed to a confocal the normals at the points of contact meet in a point; and the normals to the conic at the vertices of the triangle meet in a point.

21. Shew that

$$\frac{(x-a)^2+(y-b)^2}{r^2}=\frac{(x-a')^2+(y-b')^2}{r'^2}$$

is the circle which has as diameter the line joining the centres of similitude of the circles

$$(x-a)^2+(y-b)^2-r^2=0, \quad (x-a')^2+(y-b')^2-r'^2=0.$$

Prove that if a circle cut two circles of radii r and r' at angles α and α' respectively, it cuts the circle of similitude of the two circles at right angles if

$$r'\cos\alpha=r\cos\alpha'.$$

CHAPTER XIV.

AREAL COORDINATES.

256. Definition. Let ABC be a triangle, and P any point in its plane, then the areal coordinates of the point P with reference to the triangle ABC are defined as the three ratios

$$\frac{\triangle PBC}{\triangle ABC}, \quad \frac{\triangle PCA}{\triangle BCA}, \quad \frac{\triangle PAB}{\triangle CAB}.$$

It will be convenient to denote these by X, Y, Z respectively. It will be seen at once if the point P lie within the triangle ABC that

$$X + Y + Z = 1.$$

But this identical relation holds wherever the point P may be in the plane of the triangle ABC, for it is to be observed that the signs of the areas have to be taken into account (§ 10). Thus if P and A be on the same side of the line BC, then the areas PBC and ABC will have the same sign and the ratio of these two areas will be positive; but if P and A be on opposite sides of BC the ratio of the areas will be negative.

Thus X will be positive or negative according as P and A are on the same or opposite sides of BC.

Thus Y will be positive or negative according as P and B are on the same or opposite sides of CA.

Thus Z will be positive or negative according as P and C are on the same or opposite sides of AB.

It will be seen that the three denominators we have given for X, Y, Z are the same both in magnitude and sign, for

$$\triangle ABC = \triangle BCA = \triangle CAB,$$

the cyclical order of the letters being the same in all.

257. It is now clear that the relation $X + Y + Z = 1$ must hold wherever P be in the plane.

For let PA meet BC in D.

Thus on addition

$$\triangle PCA + \triangle PAB = \triangle PCB + \triangle ABC,$$

that is $$\triangle PAB + \triangle PCA + \triangle PBC = \triangle ABC,$$

$$\therefore X + Y + Z = 1.$$

Examples. 1. The areal coordinates of the vertices of the triangle of reference ABC are (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively.

2. The areal coordinates of the middle points of the sides of the triangle of reference are $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$.

3. The areal coordinates of the centroid of the triangle ABC are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4. The areal coordinates of the orthocentre of the triangle of reference are $(\cot B \cot C, \cot C \cot A, \cot A \cot B)$.

258. Formulae of transition from Cartesian to areal coordinates.

The following relations are of great use and importance.

If (x_1, y_1), (x_2, y_2), (x_3, y_3) *be the Cartesian coordinates referred to any axes (rectangular or oblique) in its plane of the vertices of the triangle ABC, and (x, y) the Cartesian coordinates of P in the plane, then*

$$x = x_1X + x_2Y + x_3Z,$$

$$y = y_1X + y_2Y + y_3Z,$$

where X, Y, Z are the areal coordinates of P referred to the triangle ABC.

For $X = \dfrac{\Delta PBC}{\Delta ABC} = \begin{vmatrix} x, & x_2, & x_3 \\ y, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} \div \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$,

$$\therefore\ x(y_2 - y_3) + y(x_3 - x_2) + (x_2 y_3 - x_3 y_2) = \Delta X,$$

where $$\Delta \equiv \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}.$$

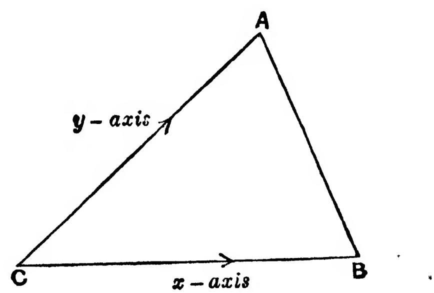
Similarly
$$x(y_3 - y_1) + y(x_1 - x_3) + (x_3 y_1 - x_1 y_3) = \Delta Y,$$
and $$x(y_1 - y_2) + y(x_2 - x_1) + (x_1 y_2 - x_2 y_1) = \Delta Z.$$

Multiplying these by x_1, x_2, x_3 respectively we get by addition
$$x\Delta = \Delta(x_1 X + x_2 Y + x_3 Z),$$
whence $$x = x_1 X + x_2 Y + x_3 Z.$$

Similarly on multiplying by y_1, y_2, y_3 we get
$$y = y_1 X + y_2 Y + y_3 Z.$$

259. The following special case of the above will be found useful.

Let the axes of the Cartesian coordinates be the sides CB, CA of the triangle ABC.



Then the coordinates of A, B, C are respectively
$$(0, b),\quad (a, 0),\quad (0, 0),$$

that is
$$x = aY,$$
$$y = bX,$$

or
$$X = \frac{y}{b}, \quad Y = \frac{x}{a},$$

and
$$Z = 1 - \frac{x}{a} - \frac{y}{b}.$$

260. Area of triangle.

From the relations of § 258 we can see that *the area of the triangle the areal coordinates of whose vertices are* (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) *is*

$$\begin{vmatrix} X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \\ X_3, & Y_3, & Z_3 \end{vmatrix} \times \triangle ABC.$$

For (x_1, y_1), (x_2, y_2), (x_3, y_3) being the Cartesian coordinates of the vertices A, B, C of the triangle of reference and ω the angle between the axes, the area will be

$$\tfrac{1}{2}\sin\omega \begin{vmatrix} x_1X_1+x_2Y_1+x_3Z_1, & x_1X_2+x_2Y_2+x_3Z_2, & x_1X_3+x_2Y_3+x_3Z_3 \\ y_1X_1+y_2Y_1+y_3Z_1, & y_1X_2+y_2Y_2+y_3Z_2, & y_1X_3+y_2Y_3+y_3Z_3 \\ X_1+Y_1+Z_1, & X_2+Y_2+Z_2, & X_3+Y_3+Z_3 \end{vmatrix},$$

and this

$$= \tfrac{1}{2}\sin\omega \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} \begin{vmatrix} X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \\ X_3, & Y_3, & Z_3 \end{vmatrix}$$

$$= \begin{vmatrix} X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \\ X_3, & Y_3, & Z_3 \end{vmatrix} \times \triangle ABC.$$

261. Distance between two points whose areal coordinates are known.

The simpler relations of § 259 enable us to obtain an expression for the distance between two points P and Q whose areal coordinates are (X_1, Y_1, Z_1), (X_2, Y_2, Z_2).

For let (x_1, y_1), (x_2, y_2) be the Cartesian coordinates of P and Q referred to CB, CA as axes; then

$$
\begin{aligned}
PQ^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos C \\
&= a^2(Y_1 - Y_2)^2 + b^2(X_1 - X_2)^2 + 2ab(Y_1 - Y_2)(X_1 - X_2)\cos C \\
&= a^2(Y_1 - Y_2)^2 + b^2(X_1 - X_2)^2 \\
&\qquad + (a^2 + b^2 - c^2)(Y_1 - Y_2)(X_1 - X_2) \\
&= a^2(Y_1 - Y_2)(Y_1 - Y_2 + X_1 - X_2) \\
&\quad + b^2(X_1 - X_2)(X_1 - X_2 + Y_1 - Y_2) - c^2(X_1 - X_2)(Y_1 - Y_2) \\
&= -a^2(Y_1 - Y_2)(Z_1 - Z_2) - b^2(Z_1 - Z_2)(X_1 - X_2) \\
&\qquad - c^2(X_1 - X_2)(Y_1 - Y_2),
\end{aligned}
$$

by using the relations

$$X_1 + Y_1 + Z_1 = 1, \qquad X_2 + Y_2 + Z_2 = 1.$$

262. Areal coordinates of a point on a line joining two given points.

If (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) be the areal coordinates of P and Q, and R be a point in PQ such that $PR : RQ = k : l$ the areal coordinates of R will be

$$\frac{kX_2 + lX_1}{k + l}, \quad \frac{kY_2 + lY_1}{k + l}, \quad \frac{kZ_2 + lZ_1}{k + l}.$$

For denoting by (x_1, y_1), (x_2, y_2) the Cartesian coordinates of P and Q referred to CB, CA as axes, the coordinates of R referred to these same axes will be (§ 9)

$$\frac{kx_2 + lx_1}{k + l}, \quad \frac{ky_2 + ly_1}{k + l},$$

that is

$$\frac{a(kY_2 + lY_1)}{k + l}, \quad \frac{b(kX_2 + lX_1)}{k + l},$$

$$\therefore X_R = \frac{y_R}{b} = \frac{kX_2 + lX_1}{k + l},$$

$$Y_R = \frac{x_R}{a} = \frac{kY_2 + lY_1}{k + l},$$

and

$$Z_R = 1 - \frac{kX_2+lX_1}{k+l} - \frac{kY_2+lY_1}{k+l} = \frac{k(1-X_2-Y_2)+l(1-X_1-Y_1)}{k+l}$$

$$= \frac{kZ_2+lZ_1}{k+l}.$$

263. The homogeneity of the equations of loci in areal coordinates.

By means of the relation $X+Y+Z=1$ the algebraical equation of any curve in areal coordinates can be made homogeneous in X, Y, Z. Thus the most general equation of the second order in X, Y, Z is of the form

$$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY$$
$$+2UX+2VY+2WZ+D=0,$$

and this is the same as

$$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY$$
$$+(X+Y+Z)(2UX+2VY+2WZ)+D(X+Y+Z)^2=0,$$

which is of the form

$$A'X^2+B'Y^2+C'Z^2+2F'YZ+2G'ZX+2H'XY=0,$$

and this is homogeneous in X, Y, Z.

Similarly equations in X, Y, Z of higher order can be made homogeneous.

264. The linear equation.

The general equation of a line in Cartesian coordinates is

$$Ax+By+C=0.$$

Using § 258 so as to transform this to areals we shall get

$$A(x_1X+x_2Y+x_3Z)+B(y_1X+y_2Y+y_3Z)+C(X+Y+Z)=0,$$

which is of the form

$$LX+MY+NZ=0.$$

Conversely it is obvious that the equation of the first order

$$LX+MY+NZ=0$$

must represent a line, for if (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) be three points on the locus represented by this equation,

$$LX_1 + MY_1 + NZ_1 = 0,$$
$$LX_2 + MY_2 + NZ_2 = 0,$$
$$LX_3 + MY_3 + NZ_3 = 0,$$
$$\therefore \begin{vmatrix} X_1, & Y_1, & Z_1 \\ X_2, & Y_2, & Z_2 \\ X_3, & Y_3, & Z_3 \end{vmatrix} = 0,$$

shewing that the area of the triangle formed by the three points is zero. These points then must be collinear.

265. The line at infinity.

The equation

$$X + Y + Z = \epsilon (lX + mY + nZ)$$

becomes on transforming to CB, CA as Cartesian axes

$$1 - \epsilon l \frac{y}{b} - \epsilon m \frac{x}{a} = \epsilon n \left(1 - \frac{x}{a} - \frac{y}{b}\right),$$

that is $$\epsilon (m - n) \frac{x}{a} + \epsilon (l - n) \frac{y}{b} = 1 - \epsilon n,$$

which is a line whose intercepts on the axes are

$$\frac{a(1 - \epsilon n)}{\epsilon (m - n)}, \quad \frac{b(1 - \epsilon n)}{\epsilon (l - n)},$$

both of which get larger and larger as ϵ gets smaller and smaller. And in the limit when ϵ approaches zero these become infinite. Thus

$$X + Y + Z = \operatorname*{Lt}_{\epsilon = 0} \epsilon (lX + mY + nZ)$$

is a line with infinite intercepts on the axes. This line we call the line at infinity. Its equation is often written

$$X + Y + Z = 0,$$

but it is misleading so to write it unless its use is properly understood, for while ϵ is very small,

$$\epsilon (lX + mY + nZ)$$

is not so, being in fact equal to $X+Y+Z$ which is unity. The justification for writing the equation in the form

$$X+Y+Z=0$$

will appear presently. (See §§ 266, 278, 286, 300 *a*.)

266. Parallel lines.

We may make use of the line at infinity to find the condition that the lines

$$lX+mY+nZ=0,$$

$$l'X+m'Y+n'Z=0$$

should be parallel.

For we regard parallel lines as lines which intersect on the line at infinity, that is on

$$X+Y+Z=\epsilon(\lambda X+\mu Y+\nu Z).$$

These three equations then must hold simultaneously,

$$\therefore \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 1-\epsilon\lambda, & 1-\epsilon\mu, & 1-\epsilon\nu \end{vmatrix}=0,$$

that is when ϵ approaches zero,

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ 1, & 1, & 1 \end{vmatrix}=0.$$

We see now that this is the result we should have obtained had we made the equations

$$lX+mY+nZ=0,$$

$$l'X+m'Y+n'Z=0,$$

$$X+Y+Z=0,$$

hold simultaneously, in other words if we had written the line at infinity

$$X+Y+Z=0.$$

Examples. 1. Obtain the equations of the lines joining the middle points of the sides of the triangle of reference in the form $Y+Z-X=0$, $Z+X-Y=0$, $X+Y-Z=0$.

Shew that these are parallel to the sides of the triangle.

2. Find the equation of the line which passes through the vertex A of the triangle of reference and through the intersection of the lines

$$lX+mY+nZ=0, \quad l'X+m'Y+n'Z=0.$$

3. Find the equation of the line through the intersection of

$$lX+mY+nZ=0 \text{ and } l'X+m'Y+n'Z=0$$

and parallel to the side BC of the triangle of reference.

4. Interpret in relation to the line $lX+mY+nZ=0$ the equation $mY+nZ=0$.

5. Prove that the condition that the lines

$$lX+mY+nZ=0, \quad l'X+m'Y+n'Z=0$$

should be at right angles is

$$a^2 ll'+b^2 mm'+c^2 nn'-(mn'+m'n)\,bc\cos A-(nl'+n'l)\,ca\cos B \\ -(lm'+l'm)\,ab\cos C=0.$$

[Transform to Cartesians with CB and CA as axes.]

6. The general equation of lines parallel to the side $X=0$ of the triangle of reference is $X=k(X+Y+Z)$.

267. *To find the length of the perpendicular from (X_1, Y_1, Z_1) on the line*

$$lX+mY+nZ=0.$$

On transforming to Cartesian axes CB, CA the equation becomes

$$l\frac{y}{b}+m\frac{x}{a}+n\left(1-\frac{x}{a}-\frac{y}{b}\right)=0.$$

The perpendicular distance of (x_1, y_1) on this is (§ 109)

$$\frac{\sin C\left\{l\frac{y_1}{b}+m\frac{x_1}{a}+n\left(1-\frac{x_1}{a}-\frac{y_1}{b}\right)\right\}}{\sqrt{\left(\frac{m-n}{a}\right)^2+\left(\frac{l-n}{b}\right)^2-\frac{2(m-n)(l-n)}{ab}\cos C}}$$

$$= \frac{ab \sin C \left\{ l \frac{y_1}{b} + m \frac{x_1}{a} + n \left(1 - \frac{x_1}{a} - \frac{y_1}{b} \right) \right\}}{\sqrt{b^2 (m-n)^2 + a^2 (l-n)^2 - (a^2 + b^2 - c^2)(m-n)(l-n)}}$$

$$= \frac{2\Delta (lX_1 + mY_1 + nZ_1)}{\sqrt{a^2 (l-n)(l-m) + b^2 (m-n)(m-l) + c^2 (n-l)(n-m)}}$$

$$= \frac{2\Delta (lX_1 + mY_1 + nZ_1)}{\sqrt{\Omega}},$$

where

$$\Omega \equiv l^2a^2 + m^2b^2 + n^2c^2 - 2mnbc \cos A - 2nlca \cos B - 2lmab \cos C.$$

268. We see then that the lengths of the perpendiculars from the vertices A, B, C of the triangle of reference on the line

$$lX + mY + nZ = 0$$

are

$$\frac{2\Delta l}{D}, \quad \frac{2\Delta m}{D}, \quad \frac{2\Delta n}{D},$$

where D stands for the denominator

$$\sqrt{a^2 (l-n)(l-m) + b^2 (m-n)(m-l) + c^2 (n-l)(n-m)}.$$

Thus l, m, n are proportional to the perpendiculars from the vertices of the triangle of reference on the line.

If p, q, r be the perpendiculars from the vertices then

$$l : m : n = p : q : r.$$

And the perpendicular from (X_1, Y_1, Z_1) on the line becomes

$$\frac{2\Delta (pX_1 + qY_1 + rZ_1)}{\sqrt{a^2 (p-q)(p-r) + b^2 (q-r)(q-p) + c^2 (r-p)(r-q)}}.$$

In particular the perpendicular from A is

$$\frac{2\Delta p}{\sqrt{a^2 (p-q)(p-r) + b^2 (q-r)(q-p) + c^2 (r-p)(r-q)}}.$$

But this $= p$. Hence we see that p, q, r must satisfy the identical relation

$$a^2 (p-q)(p-r) + b^2 (q-r)(q-p) + c^2 (r-p)(r-q) = 4\Delta^2.$$

269. These perpendiculars p, q, r from the vertices A, B, C on to the line are sometimes called the coordinates of the line, such coordinates satisfying this identical relation.

We may however for all practical purposes speak of l, m, n which are proportional to p, q, r as the coordinates of the line, but there is no identical relation satisfied by l, m, n unless they are actually *equal* to p, q, r.

270. Special form of the equation of a line.

If (X, Y, Z) *be any point on a line passing through* (X_1, Y_1, Z_1) *then*

$$\frac{X - X_1}{l} = \frac{Y - Y_1}{m} = \frac{Z - Z_1}{n} = r,$$

where r *is the algebraical distance of* (X, Y, Z) *from* (X_1, Y_1, Z_1) *and* l, m, n *are constants for the line which satisfy the relations*

$$l + m + n = 0,$$

$$a^2mn + b^2nl + c^2lm = -1.$$

For let D be the point (X_1, Y_1, Z_1) and let E (X_2, Y_2, Z_2) be some other definite point on the line.

Then if P be the point (X, Y, Z) on the line and if

$$DP : PE = k : k'$$

we know that

$$X = \frac{kX_2 + k'X_1}{k + k'}, \quad Y = \frac{kY_2 + k'Y_1}{k + k'}, \quad Z = \frac{kZ_2 + k'Z_1}{k + k'},$$

$$\therefore \ X - X_1 = \frac{k(X_2 - X_1)}{k + k'} \text{ etc.,}$$

$$\therefore \ \frac{X - X_1}{X_2 - X_1} = \frac{Y - Y_1}{Y_2 - Y_1} = \frac{Z - Z_1}{Z_2 - Z_1} = \frac{k}{k + k'} = \frac{DP}{DE} = \frac{r}{r_{12}},$$

where r is the distance DP, and r_{12} the distance DE reckoned algebraically.

Now write

$$\frac{X_2 - X_1}{r_{12}} = l, \quad \frac{Y_2 - Y_1}{r_{12}} = m, \quad \frac{Z_2 - Z_1}{r_{12}} = n,$$

and the equations of the line become

$$\frac{X - X_1}{l} = \frac{Y - Y_1}{m} = \frac{Z - Z_1}{n} = r,$$

where $\cdot$ $l + m + n = \dfrac{X_2 + Y_2 + Z_2 - (X_1 + Y_1 + Z_1)}{r_{12}} = 0,$

and $a^2mn + b^2nl + c^2lm$

$$= \frac{a^2(Y_1 - Y_2)(Z_1 - Z_2) + b^2(Z_1 - Z_2)(X_1 - X_2) + c^2(X_1 - X_2)(Y_1 - Y_2)}{r_{12}^2}$$

$= - 1.$

It is easy to see that if the line be parallel to the side BC of the triangle of reference $l = 0$ for X will be constant along the line, so that in this case

$$m + n = 0,$$

$$a^2mn = - 1,$$

So that $\qquad l : m : n = 0 : 1 : - 1.$

The student will of course understand that the l, m, n used in this form of the equation of a line are not the same as the l, m, n used when the equation of the line is written

$$lX + mY + nZ = 0.$$

271. General equation of the second degree.

As the general equation of the second degree, which we will take to be

$$AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY = 0,$$

transforms to an equation of the second degree, in x, y the corresponding Cartesian coordinates by means of § 258 or § 259, it is clear that this equation must represent a conic

272. Condition for a pair of straight lines.

If the equation represent a pair of straight lines then

$$AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY$$

must be the product of two linear factors such as

$$(LX + MY + NZ)(L'X + M'Y + N'Z),$$

that is to say

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C$$
$$\equiv (LX + MY + N)(L'X + M'Y + N'),$$

and the condition for this we have seen to be

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0.$$

273. The nature of the conic represented by the general equation in the case where this is not two straight lines can be decided by transforming to Cartesians with CB, CA as axes.

We write $X = \frac{y}{b}$, $Y = \frac{x}{a}$, $Z = 1 - \frac{x}{a} - \frac{y}{b}$,

and the equation becomes

$$A\frac{y^2}{b^2} + B\frac{x^2}{a^2} + C\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 + 2F\frac{x}{a}\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$
$$+ 2G\frac{y}{b}\left(1 - \frac{x}{a} - \frac{y}{b}\right) + 2H\frac{xy}{ab} = 0.$$

The terms of the highest degree, which determine the nature of the conic are

$$\frac{B + C - 2F}{a^2}x^2 + 2\frac{C + H - F - G}{ab}xy + \frac{C + A - 2G}{b^2}y^2.$$

274. Conditions for a circle.

If the conic be a circle the terms of the highest degree must be of the form

$$K(x^2 + 2xy\cos C + y^2).$$

Therefore

$$\frac{B + C - 2F}{a^2} : \frac{C + A - 2G}{b^2} : \frac{C + H - F - G}{ab}$$
$$= 1 : 1 : \frac{a^2 + b^2 - c^2}{2ab}.$$

That is

$$\frac{B + C - 2F}{a^2} = \frac{C + A - 2G}{b^2} = \frac{2(C + H - F - G)}{a^2 + b^2 - c^2},$$

and each of these

$$= \frac{(B+C-2F)+(C+A-2G)-2(C+H-F-G)}{a^2+b^2-(a^2+b^2-c^2)}$$

$$= \frac{A+B-2H}{c^2}.$$

Thus the conditions for a circle are

$$\frac{B+C-2F}{a^2} = \frac{C+A-2G}{b^2} = \frac{A+B-2H}{c^2}.$$

275. Discrimination of ellipse, parabola and hyperbola.

Reverting to § 273 we see that the general conic will represent an ellipse, parabola, or hyperbola according as

$$(C+H-F-G)^2 \lesseqgtr (B+C-2F)(C+A-2G),$$

that is, according as

$$F^2+G^2+H^2-2FG-2GH-2HF+2AF \\ +2BG+2CH-BC-CA-AB \lesseqgtr 0,$$

that is, according as

$$\begin{vmatrix} A, & H, & G, & 1 \\ H, & B, & F, & 1 \\ G, & F, & C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \lesseqgtr 0.$$

This discrimination can be otherwise expressed:

If we write $\quad B+C-2F=p,$

$$C+A-2G=q,$$

$$A+B-2H=r,$$

we see that $\quad 2(C+H-F-G)=p+q-r.$

Thus the conic is an ellipse, parabola, or hyperbola according as

$$(p+q-r)^2 \lesseqgtr 4pq,$$

that is $\quad p^2+q^2+r^2-2pq-2qr-2rp \lesseqgtr 0.$

276. Condition for rectangular hyperbola.

Again we see that the conic will be a rectangular hyperbola if

$$\frac{B+C-2F}{a^2}+\frac{C+A-2G}{b^2}-2\frac{C+H-F-G}{ab}\cos C=0,$$

that is $$\frac{p}{a^2}+\frac{q}{b^2}-\frac{p+q-r}{ab}\cdot\frac{a^2+b^2-c^2}{2ab}=0,$$

that is $$2pb^2+2qa^2-(p+q-r)(a^2+b^2-c^2)=0,$$

that is $p(b^2+c^2-a^2)+q(c^2+a^2-b^2)+r(a^2+b^2-c^2)=0,$

which we may write

$$\frac{p\cos A}{a}+\frac{q\cos B}{b}+\frac{r\cos C}{c}=0.$$

277. Summary.

We may then sum up these results for reference.

The conic will be a circle if

$$\frac{p}{a^2}=\frac{q}{b^2}=\frac{r}{c^2}.$$

The conic will be an ellipse, parabola, or hyperbola according as

$$\begin{vmatrix} A, & H, & G, & 1 \\ H, & B, & F, & 1 \\ G, & F, & C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} \lesseqgtr 0,$$

or as this may be written

$$p^2+q^2+r^2-2pq-2qr-2rp \lesseqgtr 0.$$

The conic will be a rectangular hyperbola if

$$\frac{p\cos A}{a}+\frac{q\cos B}{b}+\frac{r\cos C}{c}=0.$$

Should the conditions for two straight lines and for a rectangular hyperbola be *both* satisfied, then of course the equation represents a pair of straight lines at right angles, these being a particular case of a rectangular hyperbola.

278. We can discriminate the ellipse, parabola and hyperbola without any transformation to Cartesians by making use of the line at infinity. For eliminating Z between

$$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY=0 \ldots (1)$$

and
$$X+Y+Z=\epsilon(lX+mY+nZ) \ldots\ldots\ldots\ldots (2),$$

we obtain on making ϵ very small

$$AX^2+BY^2+C(X+Y)^2-2(FY+GX)(X+Y)+2HXY=0,$$

that is

$$(A+C-2G)X^2+2(C+H-F-G)XY$$
$$+(B+C-2F)Y^2=0 \ldots\ldots (3).$$

The left-hand side of this equation being the product of two linear factors of the form $\lambda X+\mu Y$, $\lambda' X+\mu' Y$, (3) represents the pair of straight lines joining the vertex C of the triangle of reference to the points of intersection of (1) and (2), that is the points of intersection of the conic with the line at infinity. Now an ellipse meets the line at infinity in two imaginary points, a hyperbola meets it in two real points (§§ 183—185), and, as is known from Pure Geometry, a parabola touches it. Thus the conic will be an ellipse, parabola, or hyperbola according as

$$(C+H-F-G)^2 \lesseqgtr (A+C-2G)(B+C-2F).$$

It will be observed here again that as we make ϵ very small after the elimination of Z the result is the same as if we had written $X+Y+Z=0$ for the line at infinity.

Examples. 1. The condition that the conic

$$AX^2+BY^2+CZ^2=0$$

should be a parabola is

$$BC+CA+AB=0.$$

2. The condition that

$$FYZ+GZX+HXY=0$$

should be a parabola is

$$\pm\sqrt{F}\pm\sqrt{G}\pm\sqrt{H}=0.$$

3. Shew that the conic

$$Y^2/b^2+Z^2/c^2=X^2/a^2$$

is an ellipse, parabola, or hyperbola according as the angle A of the triangle of reference is obtuse, right, or acute, a, b, c being the lengths of the sides of the triangle.

4. Rationalise the equation

$$\sqrt{\lambda X}+\sqrt{\mu Y}+\sqrt{\nu Z}=0$$

and shew that the conic represented by it will be a parabola if

$$\lambda+\mu+\nu=0.$$

5. Shew that the equation

$$a^2YZ+b^2ZX+c^2XY=0$$

represents a circle, as does also

$$a^2YZ+b^2ZX+c^2XY+(X+Y+Z)(lX+mY+nZ)=0.$$

6. Shew that the equation

$$p(vy-wz)^2+q(wz-ux)^2+r(ux-vy)^2=0$$

represents a pair of straight lines meeting in the point whose coordinates are proportional to $\frac{1}{u}, \frac{1}{v}, \frac{1}{w}$.

279. As throughout the rest of this chapter we shall not have occasion any more to transform our areal coordinates into Cartesian, it will be unnecessary to continue to use capital letters X, Y, Z for the areal coordinates of a point. We shall accordingly make use of x, y, z instead. We shall continue for the present to use capital letters A, B, C, etc., for the coefficients in the general equation of the conic so that there may be no fear of confusing these coefficients with the lengths of the sides of the triangle of reference.

280. Intersection of line and conic.

Let us find where the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \quad \text{..............(1)}$$

meets the conic

$$f(x, y, z)\equiv Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \text{...(2)}.$$

From (1) we have

$$x=x_1+lr, \quad y=y_1+mr, \quad z=z_1+nr.$$

Substituting in (2) we get

$$A(x_1+lr)^2+B(y_1+mr)^2+C(z_1+nr)^2+2F(y_1+mr)(z_1+nr)$$
$$+2G(z_1+nr)(x_1+lr)+2H(x_1+lr)(y_1+mr)=0,$$

that is

$$f(l, m, n)r^2+2(l\xi_1+m\eta_1+n\zeta_1)r+f(x_1, y_1, z_1)=0\ldots(3),$$

where
$$\xi_1 \equiv Ax_1+Hy_1+Gz_1,$$
$$\eta_1 = Hx_1+By_1+Fz_1,$$
$$\zeta_1 = Gx_1+Fy_1+Cz_1.$$

This quadratic equation in r gives the algebraical distances from (x_1, y_1, z_1) of the points in which the line meets the conic.

281. Equation of tangent at a given point.

If (x_1, y_1, z_1) be the given point, the line (1) of the preceding article will be a tangent to the conic if both of the roots of the quadratic equation (3) in r are zero.

This requires (i) $f(x_1, y_1, z_1)=0$,

which is satisfied since the point is on the conic;

and (ii) $l\xi_1+m\eta_1+n\zeta_1=0$.

Thus the equation of the tangent is

$$(x-x_1)\xi_1+(y-y_1)\eta_1+(z-z_1)\zeta_1=0,$$

that is
$$x\xi_1+y\eta_1+z\zeta_1=f(x_1, y_1, z_1)=0.$$

Written at length the equation of the tangent is then

$$(Ax_1+Hy_1+Gz_1)x+(Hx_1+By_1+Fz_1)y$$
$$+(Gx_1+Fy_1+Cz_1)z=0.$$

If the notation of the differential calculus be employed this is

$$x\frac{\partial f}{\partial x_1}+y\frac{\partial f}{\partial y_1}+z_1\frac{\partial f}{\partial z_1}=0,$$

where $\frac{\partial f}{\partial x_1}$ means the partial differential coefficient of $f(x_1, y_1, z_1)$ with respect to x_1, and so on.

The quantities ξ_1, η_1, ζ_1, are respectively

$$\tfrac{1}{2}\frac{\partial f}{\partial x_1},\quad \tfrac{1}{2}\frac{\partial f}{\partial y_1},\quad \tfrac{1}{2}\frac{\partial f}{\partial z_1}.$$

282. Chord of Contact and Polar.

It will be convenient to write T for the expression

$$(Ax_1 + Hy_1 + Gz_1)\,x + (Hx_1 + By_1 + Fz_1)\,y + (Gx_1 + Fy_1 + Cz_1)\,z$$

so that $T = 0$ is the equation of the tangent if (x_1, y_1, z_1) be on the curve.

If (x_1, y_1, z_1) be not on the curve, it can be proved exactly as in § 135 that $T = 0$ represents the chord of contact of tangents from (x_1, y_1, z_1) and the polar of (x_1, y_1, z_1).

It can also be proved as in § 141 that the equation of the chord whose middle point is (x_1, y_1, z_1) is $T = S_1$.

283. Condition for tangency.

We can at once find the condition that the line

$$lx + my + nz = 0 \quad \text{.....................(1)}$$

should be a tangent to the conic given by the general equation. For suppose it is a tangent at (x_1, y_1, z_1), then (1) must be identical with

$$(Ax_1 + Hy_1 + Gz_1)\,x + (Hx_1 + By_1 + Fz_1)\,y$$
$$+ (Gx_1 + Fy_1 + Cz_1)\,z = 0,$$

$$\therefore\ \frac{Ax_1 + Hy_1 + Gz_1}{l} = \frac{Hx_1 + By_1 + Fz_1}{m} = \frac{Gx_1 + Fy_1 + Cz_1}{n} = \lambda \text{ (say)}.$$

Thus
$$Ax_1 + Hy_1 + Gz_1 - l\lambda = 0,$$
$$Hx_1 + By_1 + Fz_1 - m\lambda = 0,$$
$$Gx_1 + Fy_1 + Cz_1 - n\lambda = 0.$$

Also since (x_1, y_1, z_1) satisfies (1)

$$lx_1 + my_1 + nz_1 = 0.$$

Eliminating x_1, y_1, z_1, λ we get

$$\begin{vmatrix} A, & H, & G, & l \\ H, & B, & F, & m \\ G, & F, & C, & n \\ l, & m, & n, & o \end{vmatrix} = 0,$$

which when multiplied out gives

$$A_1l^2 + B_1m^2 + C_1n^2 + 2F_1mn + 2G_1nl + 2H_1lm = 0,$$

where A_1, B_1 etc. are the minors with their proper signs of A, B, C etc. in the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}.$$

We see from (1) that the condition that the conic should touch the line at infinity $x+y+z=0$ is

$$\begin{vmatrix} A, & H, & G, & 1 \\ H, & B, & F, & 1 \\ G, & F, & C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is also the condition that the conic should be a parabola.

Thus all parabolas touch the line at infinity.

284. Equation of pair of tangents.

We can prove in exactly the same way as was done in the case of Cartesian coordinates (§ 214) that the equation of the pair of tangents from (x_1, y_1, z_1) to the conic is

$$f(x, y, z) f(x_1, y_1, z_1) = T^2.$$

For we have the general equation of conics having double contact with the given conic where the chord of contact $T = 0$ meets it, viz.

$$f(x, y, z) = \lambda T^2.$$

We now choose λ so that this passes through (x_1, y_1, z_1) and thus have the equation of the pair of tangents.

285. The equation of the director circle.

The pair of tangents from (x_1, y_1, z_1) to the conic

$$S \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

is

$$(Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy)\, S_1 = (x\xi_1 + y\eta_1 + z\zeta_1)^2.$$

Expressing the condition that these should be at right angles (§ 277) we find at once

$$\{(B + C - 2F)\, S_1 - (\eta_1 - \zeta_1)^2\} \frac{\cos A}{a}$$
$$+ \{(C + A - 2G)\, S_1 - (\zeta_1 - \xi_1)^2\} \frac{\cos B}{b}$$
$$+ \{(A + B - 2H)\, S_1 - (\xi_1 - \eta_1)^2\} \frac{\cos C}{c} = 0.$$

Thus the equation of the director circle is

$$\left(\frac{p \cos A}{a} + \frac{q \cos B}{b} + \frac{r \cos C}{c}\right) S$$
$$= \frac{\cos A}{a} (\eta - \zeta)^2 + \frac{\cos B}{b} (\zeta - \xi)^2 + \frac{\cos C}{c} (\xi - \eta)^2,$$

where ξ, η, ζ are what ξ_1, η_1, ζ_1 become when we suppress the suffixes of x, y, z, that is

$$\xi \equiv Ax + Hy + Gz,\quad \eta = Hx + By + Fz,\quad \zeta = Gx + Fy + Cz.$$

286. The centre.

Tangents at the extremities of any chord through the centre are parallel.

Thus the polar of the centre must be the line at infinity.

Hence if (x_1, y_1, z_1) be the coordinates of the centre, the line

$$(Ax_1 + Hy_1 + Gz_1)\, x + (Hx_1 + By_1 + Fz_1)\, y$$
$$+ (Gx_1 + Fy_1 + Cz_1)\, z = 0$$

must be the same as

$$x + y + z = \epsilon\, (lx + my + nz),$$

when ϵ is very small.

$$\therefore\ Ax_1 + Hy_1 + Gz_1 = Hx_1 + By_1 + Fz_1$$
$$= Gx_1 + Fy_1 + Cz_1 = \lambda \text{ (say)}.$$

We thus have

$$Ax_1 + Hy_1 + Gz_1 - \lambda = 0,$$
$$Hx_1 + By_1 + Fz_1 - \lambda = 0,$$
$$Gx_1 + Fy_1 + Cz_1 - \lambda = 0,$$

and also
$$x_1 + y_1 + z_1 - 1 = 0.$$

Eliminating x_1, y_1, z_1 we get

$$\begin{vmatrix} A, & H, & G, & \lambda \\ H, & B, & F, & \lambda \\ G, & F, & C, & \lambda \\ 1, & 1, & 1, & 1 \end{vmatrix} = 0,$$

$$\therefore \begin{vmatrix} A, & H, & G, & \lambda \\ H, & B, & F, & \lambda \\ G, & F, & C, & \lambda \\ 1, & 1, & 1, & 0 \end{vmatrix} + \begin{vmatrix} A, & H, & G, & 0 \\ H, & B, & F, & 0 \\ G, & F, & C, & 0 \\ 1, & 1, & 1, & 1 \end{vmatrix} = 0,$$

that is

$$\lambda \begin{vmatrix} A, & H, & G, & 1 \\ H, & B, & F, & 1 \\ G, & F, & C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} + \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0.$$

Thus the coordinates of the centre are given by

$$Ax_1 + Hy_1 + Gz_1 = Hx_1 + By_1 + Fz_1 = Gx_1 + Fy_1 + Cz_1$$
$$= -\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} \div \begin{vmatrix} A, & H, & G, & 1 \\ H, & B, & F, & 1 \\ G, & F, & C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

287. The coordinates of the centre can also be found very simply by using § 280. For as every line through the centre (x_1, y_1, z_1) meets the curve in two points equidistant from the centre the two roots of the quadratic equation (3) in r must be equal in magnitude and opposite in sign.

$$\therefore\ l\xi_1 + m\eta_1 + n\zeta_1 = 0,$$

for all values of l, m, n.

But
$$l + m + n = 0 \qquad (\S\ 270),$$
$$\therefore\ \xi_1 = \eta_1 = \zeta_1.$$

288. The Foci.

If (x_1, y_1, z_1) be a focus of

$$S \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

the pair of tangents from (x_1, y_1, z_1), viz.

$$(Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy)\, S_1 - (x\xi_1 + y\eta_1 + z\zeta_1)^2 = 0,$$

satisfy the conditions for a circle.

These are (§ 277),

$$\frac{(B + C - 2F)\, S_1 - (\eta_1 - \zeta_1)^2}{a^2} = \frac{(C + A - 2G)\, S_1 - (\zeta_1 - \xi_1)^2}{b^2} = \frac{(A + B - 2H)\, S_1 - (\xi_1 - \eta_1)^2}{c^2}.$$

The coordinates of the foci then are given by

$$\frac{pS - (\eta - \zeta)^2}{a^2} = \frac{qS - (\zeta - \xi)^2}{b^2} = \frac{rS - (\xi - \eta)^2}{c^2}.$$

289. The Axes.

The foci are by the last article given by

$$\frac{pS - (\eta - \zeta)^2}{a^2} = \frac{qS - (\zeta - \xi)^2}{b^2} = \frac{rS - (\xi - \eta)^2}{c^2} = \lambda \text{ (say)},$$

whence we have

$$pS - (\eta - \zeta)^2 - \lambda a^2 = 0,$$
$$qS - (\zeta - \xi)^2 - \lambda b^2 = 0,$$
$$rS - (\xi - \eta)^2 - \lambda c^2 = 0.$$

Eliminating S and λ we see that the foci lie on the conic

$$\begin{vmatrix} p, & (\eta - \zeta)^2, & a^2 \\ q, & (\zeta - \xi)^2, & b^2 \\ r, & (\xi - \eta)^2, & c^2 \end{vmatrix} = 0 \quad \text{..................(1)},$$

which is satisfied by $\xi = \eta = \zeta$, that is (§ 286) by the centre of the given conic.

Thus (1) is the equation of the axes, for only one conic can be drawn through five points of which not more than three are collinear.

I am not aware that the equation of the axes has ever been given in so simple a form before.

290. The Asymptotes.

The equation of the asymptotes will be

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + k(x+y+z)^2 = 0,$$

where k is a constant (§ 186).

But the asymptotes pass through the centre (x_1, y_1, z_1) given by

$$Ax_1 + Hy_1 + Gz_1 = Hx_1 + By_1 + Fz_1 = Gx_1 + Fy_1 + Cz_1 = \lambda,$$

where λ has the value found in § 286.

Thus

$$\begin{aligned} k &= -(Ax_1^2 + By_1^2 + Cz_1^2 + 2Fy_1z_1 + 2Gz_1x_1 + 2Hx_1y_1) \\ &= -x_1(Ax_1 + Hy_1 + Gz_1) - y_1(Hx_1 + By_1 + Fz_1) \\ &\qquad\qquad - z_1(Gx_1 + Fy_1 + Cz_1) \\ &= -\lambda(x_1 + y_1 + z_1) = -\lambda. \end{aligned}$$

Thus the asymptotes of $f(x, y, z) = 0$ are

$$f(x,y,z)\begin{vmatrix} A, & H, & G, & 1 \\ H, & B, & F, & 1 \\ G, & F, & C, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} + (x+y+z)^2 \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0.$$

291. The student can easily prove as in § 138 that the condition that the lines

$$lx + my + nz = 0,$$

$$l'x + m'y + n'z = 0,$$

should be conjugate lines for the conic,

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

is
$$\begin{vmatrix} A, & H, & G, & l \\ H, & B, & F, & m \\ G, & F, & C, & n \\ l', & m', & n' & 0 \end{vmatrix} = 0,$$
and that the condition that the pair of lines
$$ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0,$$
should be conjugate is
$$A_1u + B_1v + C_1w + 2F_1u' + 2G_1v' + 2H_1w' = 0,$$
where A_1, B_1, etc., are the minors with their proper signs of A, B, etc., in the determinant
$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}.$$

Examples. 1. If the vertex A of the triangle of reference be a focus of the conic $Ax^2+By^2+Cz^2=0$ then $a^2=b^2+c^2$ and $Bb^2=Cc^2$.

2. Shew that the equation
$$x^2+y^2+z^2-2yz+2zx+2xy=0,$$
represents a hyperbola touching the side $x=0$ of the triangle of reference at its middle point, and having for asymptotes the sides $y=0$, $z=0$.

3. The condition that
$$lx+my+nz=0 \text{ and } l'x+m'y+n'z=0,$$
should be conjugate lines for the conic
$$ux^2+vy^2+wz^2=0,$$
is
$$\frac{ll'}{u}+\frac{mm'}{v}+\frac{nn'}{w}=0.$$

4. The coordinates of the centre of the conic
$$\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0$$
are proportional to $\mu+\nu$, $\nu+\lambda$, $\lambda+\mu$.

292. Conics circumscribing the triangle of reference.

The equation of a conic circumscribing the triangle of reference is of the form
$$Fyz + Gzx + Hxy = 0.$$

For taking the conic

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

and expressing the fact that this passes through (1, 0, 0) the vertex A of the triangle of reference we get $A = 0$.

Similarly $$B = 0 = C.$$

293. Conic touching the sides of the triangle of reference.

The general equation of such conics is of the form

$$\sqrt{\lambda x} + \sqrt{\mu y} + \sqrt{\nu z} = 0.$$

For taking the general conic and putting $z = 0$, which is the equation of the side AB of the triangle of reference we obtain

$$Ax^2 + By^2 + 2Hxy = 0.$$

The left-hand side of this must be a perfect square in x and y, otherwise the conic would meet $z = 0$ in two different points.

$$\therefore\ H^2 = AB.$$

Similarly $$F^2 = BC \text{ and } G^2 = CA.$$

Thus the equation of the conic becomes

$$Ax^2 + By^2 + Cz^2 \pm 2\sqrt{BC}\,yz \pm 2\sqrt{CA}\,zx \pm 2\sqrt{AB}\,xy = 0.$$

Now put $$A = \lambda^2, \quad B = \mu^2, \quad C = \nu^2.$$

The equation becomes

$$\lambda^2x^2 + \mu^2y^2 + \nu^2z^2 \pm 2\mu\nu yz \pm 2\nu\lambda zx \pm 2\lambda\mu xy = 0.$$

But the left side is a perfect square if we take all the ambiguous signs positive or if we take two negative and one positive.

We must then exclude these cases and we have left one of the two forms

$$\lambda^2x^2 + \mu^2y^2 + \nu^2z^2 - 2\mu\nu yz - 2\nu\lambda zx - 2\lambda\mu xy = 0,$$

$$\lambda^2x^2 + \mu^2y^2 + \nu^2z^2 + 2\mu\nu yz + 2\nu\lambda zx - 2\lambda\mu xy = 0.$$

But if we write $-\nu'$ for ν in the second of these it becomes the same form as the first, which is then the most general form.

It can be written

$$(\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z})(-\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z})(\sqrt{\lambda x}-\sqrt{\mu y}+\sqrt{\nu z})$$
$$(\sqrt{\lambda x}+\sqrt{\mu y}-\sqrt{\nu z})=0.$$

Thus $$\sqrt{\lambda x}\pm\sqrt{\mu y}\pm\sqrt{\nu z}=0,$$

which we may write

$$\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0,$$

the ambiguous sign being understood before each of the radicals.

294. *The condition that*

$$lx+my+nz=0$$

should touch the conic

$$\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0$$

is $$\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}=0.$$

For if we eliminate z between the two equations we get

$$(\sqrt{\lambda x}+\sqrt{\mu y})^2=-\nu\times\frac{lx+my}{n},$$

that is

$$\lambda x+\mu y+2\sqrt{\lambda\mu xy}=\frac{-\nu(lx+my)}{n},$$

that is

$$x\left(\lambda+\frac{l\nu}{n}\right)+2\sqrt{\lambda\mu xy}+\left(\mu+\frac{\nu m}{n}\right)y=0.$$

The left side must be a perfect square in $\sqrt{x}$ and $\sqrt{y}$,

$$\therefore\ \lambda\mu=\left(\lambda+\frac{l\nu}{n}\right)\left(\mu+\frac{\nu m}{n}\right),$$

which reduces to

$$\lambda mn+\mu nl+\nu lm=0,$$

that is $$\frac{\lambda}{l}+\frac{\mu}{m}+\frac{\nu}{n}=0.$$

295. Conics for which the triangle of reference is self-polar.

The general equation of conics for which the triangle of reference is self-polar, that is to say each vertex is the pole of the opposite side is

$$Ax^2 + By^2 + Cz^2 = 0.$$

For the polar of the vertex A (1, 0, 0) for

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

is
$$Ax + Hy + Gz = 0,$$

and this has to reduce to $x = 0$;

$$\therefore \quad G = 0 \text{ and } H = 0.$$

Similarly $F = 0$.

The condition that $lx + my + nz = 0$ should be a tangent is

$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = 0.$$

For the tangent at (x_1, y_1, z_1) is

$$Axx_1 + Byy_1 + Czz_1 = 0.$$

Making the two lines identical we have

$$\frac{Ax_1}{l} = \frac{By_1}{m} = \frac{Cz_1}{n}.$$

But
$$Ax_1^2 + By_1^2 + Cz_1^2 = 0,$$

$$\therefore \quad \frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = 0.$$

296. Equation of the circumcircle of the triangle of reference.

The general equation of conics circumscribing the triangle of reference is

$$Fyz + Gzx + Hxy = 0.$$

The conditions for a circle become in this case

$$\frac{F}{a^2} = \frac{G}{b^2} = \frac{H}{c^2}.$$

Thus the circumcircle has for equation

$$a^2yz + b^2zx + c^2xy = 0,$$

or, as it may be written,

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 0.$$

297. Equations of incircle and ecircles of the triangle of reference.

The equation of a conic touching the sides of the triangle is given by

$$\lambda^2x^2 + \mu^2y^2 + \nu^2z^2 - 2\mu\nu yz - 2\nu\lambda zx - 2\lambda\mu xy = 0.$$

Expressing the conditions for a circle we have

$$\frac{\mu^2 + \nu^2 + 2\mu\nu}{a^2} = \frac{\nu^2 + \lambda^2 + 2\nu\lambda}{b^2} = \frac{\lambda^2 + \mu^2 + 2\lambda\mu}{c^2}$$

$$\therefore \quad \frac{\mu + \nu}{\pm a} = \frac{\nu + \lambda}{\pm b} = \frac{\lambda + \mu}{\pm c},$$

that is either $\quad \dfrac{\mu + \nu}{a} = \dfrac{\nu + \lambda}{b} = \dfrac{\lambda + \mu}{c},$

which give $\quad \dfrac{\lambda}{b + c - a} = \dfrac{\mu}{c + a - b} = \dfrac{\nu}{a + b - c},$

or $\quad \dfrac{\mu + \nu}{a} = \dfrac{\nu + \lambda}{b} = \dfrac{\lambda + \mu}{-c},$

which give $\quad \dfrac{\lambda}{b - c - a} = \dfrac{\mu}{-c + a - b} = \dfrac{\nu}{a + b + c},$

or $\quad \dfrac{\mu + \nu}{a} = \dfrac{\nu + \lambda}{-b} = \dfrac{\lambda + \mu}{c},$

which give $\quad \dfrac{\lambda}{-b + c - a} = \dfrac{\mu}{c + a + b} = \dfrac{\nu}{a - b - c}.$

or $\quad \dfrac{\mu + \nu}{a} = \dfrac{\nu + \lambda}{-b} = \dfrac{\lambda + \mu}{-c},$

which give $\quad \dfrac{\lambda}{-b - c - a} = \dfrac{\mu}{-c + a + b} = \dfrac{\nu}{a - b + c}.$

Whence we get as the equations of the four circles touching the sides of the triangle of reference

$$\sqrt{(s-a)x} + \sqrt{(s-b)y} + \sqrt{(s-c)z} = 0,$$
$$\sqrt{(s-b)x} + \sqrt{(s-a)y} + \sqrt{-sz} = 0,$$
$$\sqrt{(s-c)x} + \sqrt{-sy} + \sqrt{(s-a)z} = 0,$$
$$\sqrt{-sx} + \sqrt{(s-c)y} + \sqrt{(s-b)z} = 0.$$

It is easy to see that the first of these gives the incircle, the second the ecircle opposite to C and so on, for the z coordinate of every point on this ecircle is negative, whereas the x and y coordinates are positive.

We observe that the equation of the ecircle opposite to A is got from that of the incircle by writing $-a$ for a.

298. Equation of the nine points circle of the triangle of reference.

This circle has to pass through the middle points of the sides and the coordinates of these points are $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, 0)$.

Taking the general equation

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

we have the conditions

$$B + C + 2F = 0,$$
$$C + A + 2G = 0,$$
$$A + B + 2H = 0.$$

And the conditions for a circle are

$$\frac{B+C-2F}{a^2} = \frac{C+A-2G}{b^2} = \frac{A+B-2H}{c^2}.$$

$$\therefore \frac{B+C}{a^2} = \frac{C+A}{b^2} = \frac{A+B}{c^2},$$

which give $\dfrac{A}{b^2+c^2-a^2} = \dfrac{B}{c^2+a^2-b^2} = \dfrac{C}{a^2+b^2-c^2} = k$ (say).

$$\therefore 2F = -(B+C) = -k(2a^2); \quad \therefore F = -ka^2.$$

Hence the equation of the nine points circle is

$$(b^2+c^2-a^2)\,x^2+(c^2+a^2-b^2)\,y^2+(a^2+b^2-c^2)\,z^2 \\ -2a^2yz-2b^2zx-2c^2xy=0,$$

that is

$$a^2\{(y-z)^2-x^2\}+b^2\{(z-x)^2-y^2\}+c^2\{(x-y)^2-z^2\}=0,$$

or $$a^2(y-z-x)(y-z+x)+b^2(z-x-y)(z-x+y) \\ +c^2(x-y-z)(x-y+z)=0,$$

which can be written

$$\frac{a^2}{y+z-x}+\frac{b^2}{z+x-y}+\frac{c^2}{x+y-z}=0,$$

299. On the form of the equation of the nine points circle.

It may have already occurred to the reader that, as the equation of the circumcircle of the triangle of reference is

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}=0,$$

and that of the nine points circle is

$$\frac{a^2}{y+z-x}+\frac{b^2}{z+x-y}+\frac{c^2}{x+y-z}=0,$$

there is some connection between the two. Such indeed is the case.

Let A', B', C' be the middle points of the sides of the triangle of reference. A' being opposite to A, B' to B and C' to C.

Let (x, y, z) be the areal coordinates of a point P referred to the triangle ABC. And let (x', y', z') be the areal coordinates of the same point referred to $A'B'C'$.

Then $$x'=\frac{\Delta PB'C'}{\Delta A'B'C'}=\frac{4\Delta PB'C'}{\Delta ABC}.$$

But the coordinates of P, B', C' referred to ABC being respectively

$$(x, y, z),\quad (\tfrac{1}{2}, 0, \tfrac{1}{2}),\quad (\tfrac{1}{2}, \tfrac{1}{2}, 0),$$

we have
$$\frac{\Delta PB'C'}{\Delta ABC}=\begin{vmatrix} x, & \frac{1}{2}, & \frac{1}{2} \\ y, & 0, & \frac{1}{2} \\ z, & \frac{1}{2}, & 0 \end{vmatrix}=\tfrac{1}{4}(y+z-x),$$

$$\therefore\ x'=y+z-x.$$

Similarly $y'=z+x-y$ and $z'=x+y-z$.

Now the nine points circle of the original triangle ABC is the circumcircle of the triangle $A'B'C'$.

If then the point P be on the nine points circle of ABC, it is on the circumcircle of $A'B'C'$.

$$\therefore\ \frac{\left(\frac{a}{2}\right)^2}{x'}+\frac{\left(\frac{b}{2}\right)^2}{y'}+\frac{\left(\frac{c}{2}\right)^2}{z'}=0,$$

that is
$$\frac{a^2}{y+z-x}+\frac{b^2}{z+x-y}+\frac{c^2}{x+y-z}=0.$$

Again the equation of the incircle of $A'B'C'$ referred to $A'B'C'$ would be

$$\sqrt{(s'-a')\,x'}+\sqrt{(s'-b')\,y'}+\sqrt{(s'-c')\,z'}=0,$$

where a', b', c' refer to $A'B'C'$. And $a':b':c'=a:b:c$.

Therefore the incircle of $A'B'C'$ has for its equation referred to ABC

$$\sqrt{(s-a)(y+z-x)}+\sqrt{(s-b)(z+x-y)}+\sqrt{(s-c)(x+y-z)}=0.$$

We see then how when we know the equation of some locus connected with the triangle ABC we can find the equation of the corresponding locus connected with $A'B'C'$.

300. The Circular Points at infinity.

We have seen that the equation

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0\ \ldots\ldots(1)$$

represents a circle if

$$\frac{B+C-2F}{a^2}=\frac{C+A-2G}{b^2}=\frac{A+B-2H}{c^2}.$$

Suppose these conditions fulfilled and equate each of these fractions to λ.

$$\therefore\ 2F = B + C - \lambda a^2,$$
$$2G = C + A - \lambda b^2,$$
$$2H = A + B - \lambda c^2.$$

Thus the equation of the circle becomes

$$Ax^2 + By^2 + Cz^2 + (B + C - \lambda a^2)\,yz + (C + A - \lambda b^2)\,zx + (A + B - \lambda c^2)\,xy = 0,$$

that is

$$(Ax + By + Cz)(x + y + z) - \lambda(a^2yz + b^2zx + c^2xy) = 0 \ldots (2).$$

Thus points common to this circle and the circumcircle of the triangle of reference satisfy

$$(Ax + By + Cz)(x + y + z) = 0.$$

We see then that $Ax + By + Cz = 0$ is the radical axis of (1) and the circumcircle; and further that (1) meets the line at infinity, $x + y + z = 0$, in the same points in which the circumcircle meets it.

Thus all circles in a plane go through the same two points on the line at infinity, viz. the two points in which the line at infinity cuts the circumcircle of the triangle of reference.

These points are called 'the circular points at infinity.' For their use see my *Course of Pure Geometry*, Chapter XVIII.

300 *a*. It seems desirable here again to justify our taking the equation of the line at infinity as

$$x + y + z = 0,$$

instead of using its more accurate expression

$$x + y + z = \text{Lt}\,\epsilon\,(lx + my + nz) \ldots\ldots\ldots\ldots\ldots (3).$$

This line meets the circle (2) in points determined by

$$\epsilon\,(lx + my + nz)(Ax + By + Cz) - \lambda(a^2yz + b^2zx + c^2xy) = 0,$$

that is by

$$\epsilon\,(lx + my + nz) = \frac{\lambda(a^2yz + b^2zx + c^2xy)}{Ax + By + Cz}.$$

Now the points satisfying this and (3) are at infinity, and so the numerator of the right-hand side unless it be zero will be large compared with the denominator, whereas the left-hand side is finite. Hence we must have

$$a^2yz + b^2zx + c^2xy = 0.$$

Thus the circle meets the line at infinity in the same points in which the circumcircle meets it.

Another way of regarding the matter is this. We see that $x + y + z$ is always finite, being in fact equal to unity, and the equation (2) can be written

$$Ax + By + Cz - \lambda(a^2yz + b^2zx + c^2xy) = 0,$$

so that at infinity the terms of the higher order in x, y, z, namely

$$\lambda(a^2yz + b^2zx + c^2xy),$$

must overwhelm those of the lower order, namely

$$Ax + By + Cz,$$

unless indeed $\qquad a^2yz + b^2zx + c^2xy = 0.$

This is just what we get when in (2) we put $x + y + z = 0$.

Thus when we write $x + y + z = 0$ we are really expressing the fact that even when x, y and z have infinite values they assume these values consistently with $x + y + z$ being finite; and if $x + y + z$ be finite it is the same as if we write

$$x + y + z = 0,$$

for finite quantities are negligible in comparison with infinite ones.

301. Radical axis of two circles.

If $\quad S \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \ldots(1),$

$$S' \equiv A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0 \ldots(2),$$

be two circles, and if

$$\lambda = \frac{B + C - 2F}{a^2} = \frac{C + A - 2G}{b^2} = \frac{A + B - 2H}{c^2},$$

and $\quad \lambda' = \dfrac{B' + C' - 2F'}{a^2} = \dfrac{C' + A' - 2G'}{b^2} = \dfrac{A' + B' - 2H'}{c^2},$

the equations can be written (§ 300)

$$(Ax+By+Cz)(x+y+z)-\lambda(a^2yz+b^2zx+c^2xy)=0,$$

$$(A'x+B'y+C'z)(x+y+z)-\lambda'(a^2yz+b^2zx+c^2xy)=0.$$

Thus points common to the two circles satisfy

$$\frac{(Ax+By+Cz)(x+y+z)}{\lambda}=\frac{(A'x+B'y+C'z)(x+y+z)}{\lambda'}.$$

Thus the radical axis of the two circles is

$$\frac{Ax+By+Cz}{\lambda}=\frac{A'x+B'y+C'z}{\lambda'}.$$

The student will have no difficulty in proving for himself that the square of the tangent from (x, y, z) to the circle (1) is $\frac{S}{\lambda}$.

Example. If p, q, r be the lengths of the tangents from the vertices A, B, C of the triangle of reference to a circle the equation of that circle is

$$(p^2x+q^2y+r^2z)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$$ [Wolstenholme].

302. Feuerbach's theorem.

Let us find the radical axis of the nine points circle of the triangle of reference, viz.

$$(b^2+c^2-a^2)x^2+(c^2+a^2-b^2)y^2+(a^2+b^2-c^2)z^2 -2a^2yz-2b^2zx-2c^2xy=0$$

and the incircle, viz.

$$(s-a)^2x^2+(s-b)^2y^2+(s-c)^2z^2-2(s-b)(s-c)yz -2(s-c)(s-a)zx-2(s-a)(s-b)xy=0.$$

Here $\lambda=\dfrac{(c^2+a^2-b^2)+(a^2+b^2-c^2)+2a^2}{a^2}=4,$

and $\lambda'=\dfrac{(s-b)^2+(s-c)^2+2(s-b)(s-c)}{a^2}=\dfrac{(2s-b-c)^2}{a^2}=1.$

Thus the radical axis of these circles is

$$\frac{(b^2+c^2-a^2)x+(c^2+a^2-b^2)y+(a^2+b^2-c^2)z}{4} =(s-a)^2x+(s-b)^2y+(s-c)^2z,$$

that is $$[(b^2+c^2-a^2)-(b+c-a)^2]\,x+\text{etc.}=0,$$
which reduces to
$$(a-b)(c-a)\,x+(b-c)(a-b)\,y+(c-a)(b-c)\,z=0,$$
that is $$\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0.$$

Now this line touches the incircle; for the condition for this is (§ 294)
$$\frac{s-a}{\dfrac{1}{b-c}}+\frac{s-b}{\dfrac{1}{c-a}}+\frac{s-c}{\dfrac{1}{a-b}}=0,$$
that is $(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)=0$,
which is satisfied.

Hence the nine points circle touches the incircle.

As the equation of the ecircle opposite to A can be derived from that of the incircle by writing $-a$ for a, and the equation of the nine points circle is unaltered by this, it is clear that this ecircle must also touch the nine points circle, and similarly the other ecircles touch it too.

This is Feuerbach's well known theorem.

303. *The condition that the circle*
$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \;\ldots\ldots(1)$$
should cut orthogonally the circumcircle of the triangle of reference is
$$Aa\cos\alpha+Bb\cos\beta+Cc\cos\gamma-\lambda abc=0,$$
where α, β, γ are the angles of the triangle and
$$\lambda=\frac{B+C-2F}{a^2}=\text{etc.}$$

For if the circles cut orthogonally the pole, with respect to either circle, of their radical axis must be at the centre of the other circle.

Now the radical axis of the two circles is (§ 300)
$$Ax+By+Cz=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let (x_1, y_1, z_1) be the pole of this with regard to (1),

$$\therefore \frac{Ax_1 + Hy_1 + Gz_1}{A} = \frac{Hx_1 + By_1 + Fz_1}{B} = \frac{Gx_1 + Fy_1 + Cz_1}{C}.$$

And if the circles cut orthogonally (x_1, y_1, z_1) must be the centre of the circumcircle.

$$\therefore x_1 : y_1 : z_1 = a \cos \alpha : b \cos \beta : c \cos \gamma.$$

$$\therefore \frac{Aa \cos \alpha + Hb \cos \beta + Gc \cos \gamma}{A} = \frac{Ha \cos \alpha + Bb \cos \beta + Fc \cos \gamma}{B}$$

$$= \frac{Ga \cos \alpha + Fb \cos \beta + Cc \cos \gamma}{C} \quad \text{...(3).}$$

Taking the first of these equalities we get

$$Aa \cos \alpha (B - H) + Bb \cos \beta (H - A) + c \cos \gamma (BG - AF) = 0.$$

We now make use of the relations

$$B + C - \lambda a^2 = 2F, \quad C + A - \lambda b^2 = 2G, \quad A + B - \lambda c^2 = 2H,$$

and so obtain on elimination of F, G, H

$$Aa \cos \alpha (B - A + \lambda c^2) + Bb \cos \beta (B - A - \lambda c^2)$$
$$+ c \cos \gamma \{B (C + A - \lambda b^2) - A (B + C - \lambda a^2)\} = 0,$$

that is

$$Aa \cos \alpha (B - A) + Bb \cos \beta (B - A) + Cc \cos \gamma (B - A)$$
$$+ \lambda Aac (c \cos \alpha + a \cos \gamma) - \lambda Bbc (c \cos \beta + b \cos \gamma) = 0,$$

which gives either $B - A = 0$ or

$$Aa \cos \alpha + Bb \cos \beta + Cc \cos \gamma - \lambda abc = 0 \quad \text{......(4).}$$

Had we taken the second of the two equalities (3) we should have obtained either (4) or $B = C$.

Thus either (4) holds or $A = B = C$. But in this latter case the radical axis becomes $x + y + z = 0$, that is the line at infinity. In other words the two circles would be concentric.

We see then that (4) is the condition that the circles should cut orthogonally.

EXAMPLES.

1. Shew that every conic circumscribing the triangle of reference and passing through the orthocentre must be a rectangular hyperbola.

2. If a rectangular hyperbola circumscribe a triangle it passes through the orthocentre.

3. The circle for which the triangle of reference is self-polar is coaxial with the circumcircle and the nine points circle.

4. The centre of a rectangular hyperbola circumscribing a triangle lies on the nine points circle.

5. The areal coordinates of the points of contact of the conic

$$\sqrt{\lambda x} + \sqrt{\mu y} + \sqrt{\nu z} = 0$$

with the sides of the triangle of reference are

$$\left(0, \frac{\nu}{\mu+\nu}, \frac{\mu}{\mu+\nu}\right), \quad \left(\frac{\nu}{\nu+\lambda}, 0, \frac{\lambda}{\nu+\lambda}\right), \quad \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu}, 0\right)$$

which become

$$\left(0, -\frac{\nu}{\lambda}, -\frac{\mu}{\lambda}\right), \quad \left(-\frac{\nu}{\mu}, 0, -\frac{\lambda}{\mu}\right), \quad \left(-\frac{\mu}{\nu}, -\frac{\lambda}{\nu}, 0\right),$$

where the conic is a parabola.

The area of the triangle formed by three tangents to a parabola is half that of the triangle formed by their points of contact.

6. The condition that the line $lx + my + nz = 0$, should touch the conic $Fyz + Gzx + Hxy = 0$ is

$$\sqrt{lF} \pm \sqrt{mG} \pm \sqrt{nH} = 0.$$

7. If the conic $Ax^2 + By^2 + Cz^2 = 0$ be a parabola, the coordinates of its focus are proportional to

$$Bb^2 + Cc^2, \quad Cc^2 + Aa^2, \quad Aa^2 + Bb^2,$$

and the equation of its axis is

$$\begin{vmatrix} x, & y, & z \\ Bb^2 + Cc^2, & Cc^2 + Aa^2, & Aa^2 + Bb^2 \\ \frac{1}{A}, & \frac{1}{B}, & \frac{1}{C} \end{vmatrix} = 0.$$

[As in § 288 the coordinates of the foci are given by

$$\frac{(B+C)(Ax^2+By^2+Cz^2)-(By-Cz)^2}{a^2} = \text{two similar expressions.}$$

Using the fact that $BC + CA + AB = 0$ in our case we have

$$\frac{BC\,(y+z-x)(x+y+z)}{a^2} = \frac{CA\,(z+x-y)(x+y+z)}{b^2}$$

$$= \frac{AB\,(x+y-z)(x+y+z)}{c^2}.$$

Dividing out by $x+y+z$ we get

$$\frac{y+z-x}{Aa^2} = \frac{z+x-y}{Bb^2} = \frac{x+y-z}{Cc^2},$$

$$\therefore \quad \frac{x}{Bb^2+Cc^2} = \frac{y}{Cc^2+Aa^2} = \frac{z}{Aa^2+Bb^2}.$$

If $lx + my + nz = 0$ be the axis, since this passes through the focus we have

$$(Bb + Cc^2)\,l + (Cc^2 + Aa^2)\,m + (Aa^2 + Bb^2)\,n = 0.$$

Also the axis goes through the centre the coordinates of which (x_1, y_1, z_1) are infinite but are in a finite ratio to one another given by (§ 286)

$$Ax_1 = By_1 = Cz_1.$$

Thus $$\frac{l}{A} + \frac{m}{B} + \frac{n}{C} = 0.$$

On eliminating l, m, n we have the equation of the axis.]

8. Obtain the equation of the director circle of the conic

$$Fyz + Gzx + Hxy = 0,$$

and deduce that the equation of the directrix in the case where the conic is a parabola is

$$\Sigma\,\frac{F\cos A}{a}\,\{(G+H-F)\,x + F\,(y+z)\} = 0.$$

9. The centre of the circle circumscribing a triangle which is self-conjugate with regard to a parabola lies on the directrix of the parabola.

10. Prove Gaskin's theorem, of which Ex. 9 is a special case; The circle circumscribing a triangle which is self-conjugate for a conic is orthogonal to the director circle of the conic. [Use § 303.]

11. Obtain the equation of the parabola touching the three sides of the triangle of reference, the point of contact with $x=0$ being the middle point of that side.

$[4x^2+y^2+z^2-2yz+4zx+4xy=0.]$

12. If in the last Example D, E, F be the points of contact with the sides, prove that EF is parallel to BC, and that AD passes through the middle point of EF. Prove further that the middle point of all chords parallel to BC lie on the line AD.

13. Shew that the equation $x^2=2kyz$ represents a conic touching the sides AB, AC of the triangle of reference in B and C. Prove that the straight line joining A to the middle point of BC cuts the conic in a point the tangent at which is parallel to BC. Further shew that this line passes through the centre of the conic.

14. Prove that if the conic $\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0$ be a parabola, the equation of its directrix is

$$\lambda(b^2+c^2-a^2)x+\mu(c^2+a^2-b^2)y+\nu(a^2+b^2-c^2)z=0.$$

Shew also that the coordinates of the focus are proportional to

$$\left(\frac{a^2}{\lambda},\ \frac{b^2}{\mu},\ \frac{c^2}{\nu}\right),$$

that the focus lies on the circumcircle of the triangle of reference, and that the directrix passes through the orthocentre of the triangle.

15. Shew that the equation of the axis of the parabola in the previous question is

$$x\left(\frac{b^2\nu}{\mu}-\frac{c^2\mu}{\nu}\right)+y\left(\frac{c^2\lambda}{\nu}-\frac{a^2\nu}{\lambda}\right)+z\left(\frac{a^2\mu}{\lambda}-\frac{b^2\lambda}{\mu}\right)=0.$$

16. The necessary and sufficient condition that the triangle $A'B'C'$ whose vertices are (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3) should be in perspective with the triangle of reference ABC (A' corresponding with A and so on) is

$$x_2y_3z_1=x_3y_1z_2.$$

17. If two triangles ABC, $A'B'C'$ be reciprocal for a conic (that is A the pole of $B'C'$, B of $C'A'$, C of $A'B'$) they will be in perspective.

18. Shew that if the sides of the pedal triangle of the triangle ABC be produced to meet the opposite sides in D, E, F, the straight line DEF is the radical axis of the circumcircle of ABC and the circle with respect to which the triangle ABC is self-conjugate.

19. The locus of the centre of a conic which touches the sides of a given triangle and passes through a given point is a conic inscribed in the triangle formed by the lines joining the middle points of the sides of the given triangle.

20. A conic passes through the vertices of the triangle of reference and their centre of mean position. One of its axes is parallel to $x=0$, the coordinates being areal. Shew that its equation is

$$\frac{a}{x}=\frac{c\cos B}{y}+\frac{b\cos C}{z}.$$

21. Shew that the locus of the centres of conics which circumscribe a given triangle ABC and have a common tangent at A is a conic.

22. O is a point whose areal coordinates are (x, y, z) with reference to the triangle ABC, whose sides are of lengths a, b, c; if P be any other point prove

$$x\,.\,PA^2+y\,.\,PB^2+z\,.\,PC^2=PO^2+a^2yz+b^2zx+c^2xy$$

and deduce the equation of a circle whose centre and radius are given.

23. The general equation of conics passing through the middle points of the sides of the triangle of reference is

$$F(z+x-y)(x+y-z)+G(x+y-z)(y+z-x)$$
$$+H(y+z-x)(z+x-y)=0.$$

24. The angular points A, B, C of a triangle are joined to any point O, and OA, OB, OC meet the opposite sides in α, β, γ. Shew that if the conic through α, β, γ and the middle points of the sides be a rectangular hyperbola, then O lies on the circle round ABC.

25. A triangle circumscribes a conic and α, β, γ are the points of contact. Shew that the intersections of the lines BC, $\beta\gamma$; CA, $\gamma\alpha$; AB, $\alpha\beta$ are collinear and that if the conic touches a fourth fixed line, this line of collinearity passes through a fixed point.

26. The locus of the pole of the line $\lambda x + \mu y + \nu z = 0$ with respect to the system of parabolas which pass through the vertices of the triangle of reference is the curve

$$\sqrt{x(\mu y + \nu z - \lambda x)} + \sqrt{y(\nu z + \lambda x - \mu y)} + \sqrt{z(\lambda x + \mu y - \nu z)} = 0.$$

27. Conics touch the sides of a triangle ABC; and the point of contact with the side BC is a fixed point. From another fixed point in BC tangents are drawn to the conics. Shew that their points of contact lie on a fixed line through A.

28. The equation of the line containing the centroid, the orthocentre, the circumcentre and the nine points centre of the triangle of reference ABC is

$$\frac{b^2 - c^2}{a} \cos A \,.\, x + \frac{c^2 - a^2}{b} \cos B \,.\, y + \frac{a^2 - b^2}{c} \cos C \,.\, z = 0.$$

Shew that this line contains the four corresponding points for the triangle joining the middle points of the sides.

29. The areal coordinates of the centre of the conic

$$\sqrt{\lambda x} + \sqrt{\mu y} + \sqrt{\nu z} = 0$$

inscribed in the triangle of reference ABC with respect to the triangle DEF joining the middle points of the sides are in the ratio

$$\lambda : \mu : \nu.$$

30. Shew that the common chord of the conic $yz + zx + xy = 0$ and its circle of curvature at the vertex A of the triangle of reference is

$$y(a^2 - c^2) + z(a^2 - b^2) = 0,$$

a, b, c being the lengths of the sides.

31. The locus of the centre of the conic $lyz + mzx + nxy = 0$, which passes through $(x'y'z')$ is a conic whose centre is at the point

$$\left(\frac{1 + x'}{4}, \quad \frac{1 + y'}{4}, \quad \frac{1 + z'}{4}\right).$$

32. The axis of a parabola is $\lambda x + \mu y + \nu z = 0$, and the tangent at the vertex is $\lambda' x + \mu' y + \nu' z = 0$, shew that its equation is of the form

$$(\lambda x + \mu y + \nu z)^2 = k(x + y + z)(\lambda' x + \mu' y + \nu' z).$$

33. A conic circumscribes a triangle and its centre moves along a median, prove that the asymptotes touch a conic which touches two of the sides of the triangle at the extremities of the remaining side.

34. A conic is inscribed in a triangle ABC and one of its asymptotes passes through a fixed point. Find the locus of the centre, and prove that if the point coincides with A the locus becomes the sides AB, AC and the straight line joining the middle points of these sides.

35. Prove that if a parabola touch the sides of the triangle ABC, the polar with respect to the parabola of the centroid G of the triangle will touch the conic which passes through A, B, C and has its centre at G.

36. Prove that if

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

represent a circle in areal coordinates its radius is

$$2R\Delta^{\frac{1}{2}} / \{(2f - b - c)(2g - c - a)(2h - a - b)\}^{\frac{1}{2}},$$

where Δ is the discriminant, and R the radius of the circumcircle of the triangle of reference.

37. Shew that one asymptote of the conic $ax^2 + by^2 + cz^2 = 0$ will be parallel to one asymptote of the conic $a'x^2 + b'y^2 + c'z^2 = 0$, if

$$\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu - 2\nu\lambda - 2\lambda\mu = 0,$$

where $\lambda \equiv bc' - b'c, \quad \mu \equiv ca' - c'a, \quad \nu \equiv ab' - a'b,$

the coordinates being areal.

38. Shew that if

$$\xi \equiv vy - wz, \quad \eta \equiv wz - ux, \quad \zeta \equiv ux - vy,$$

the asymptotes of the conic $ux^2 + vy^2 + wz^2 = 0$ are

$$u\xi^2 + v\eta^2 + w\zeta^2 = 0,$$

and that $p\xi^2 + q\eta^2 + r\zeta^2 = 0$ represents a pair of conjugate diameters provided that

$$p(v + w) + q(w + u) + r(u + v) = 0.$$

39. Shew that the equation of a pair of conjugate diameters of the conic $yz + zx + xy = 0$, may be written

$$(q - r)(y - z)^2 + (r - p)(z - x)^2 + (p - q)(x - y)^2 = 0.$$

40. Shew that the locus of the centres of all conics of given eccentricity which circumscribe a triangle is in general a curve of the fourth degree passing through the middle points of the sides of the triangle.

41. The area of the ellipse whose areal equation is

$$\lambda yz + \mu zx + \nu xy = 0,$$

bears to the area of the triangle of reference the ratio

$$4\pi\lambda\mu\nu : (2\mu\nu + 2\nu\lambda + 2\lambda\mu - \lambda^2 - \mu^2 - \nu^2)^{\frac{3}{2}}.$$

42. The areal coordinates of the point of contact of the incircle and nine-points circle of the triangle of reference are in the ratio

$$(b - c)^2 (s - a) : (c - a)^2 (s - b) : (a - b)^2 (s - c).$$

CHAPTER XV.

HOMOGENEOUS COORDINATES IN GENERAL.

304. Areal coordinates, which we have treated in the last chapter, are only a particular case of a general system of homogeneous coordinates which we now proceed to explain. We transform the Cartesian coordinates (x, y) referred to any axes, rectangular or oblique, in the plane to new coordinates (X, Y, Z) by the relations

$$x = \lambda_1 X + \mu_1 Y + \nu_1 Z \quad\text{.....................}(1),$$

$$y = \lambda_2 X + \mu_2 Y + \nu_2 Z \quad\text{.....................}(2),$$

where X, Y, Z are connected by a linear relation

$$1 = \alpha X + \beta Y + \gamma Z \quad\text{.....................}(3).$$

As yet no geometrical meaning is given to the new coordinates X, Y, Z.

We can solve the above equations and express X, Y, Z in terms of x and y, provided the determinant

$$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \alpha, & \beta, & \gamma \end{vmatrix} \text{ be not zero.}$$

We will write these results

$$X = A_1 x + B_1 y + C_1,$$

$$Y = A_2 x + B_2 y + C_2,$$

$$Z = A_3 x + B_3 y + C_3.$$

It is clear that the equation $lX + mY + nZ = 0$, being equivalent to

$$l(A_1 x + B_1 y + C_1) + m(A_2 x + B_2 y + C_2) + n(A_3 x + B_3 y + C_3) = 0$$

represents a straight line.

And in particular $X=0$, $Y=0$, $Z=0$ represent straight lines.

These last three lines form a triangle ABC which we shall speak of as the *triangle of reference*.

The vertex A of the triangle being given by $Y=0$, $Z=0$, we have from (3) $X=1/\alpha$.

Thus the coordinates of A are $(1/\alpha, 0, 0)$. Similarly, those of B and C are $(0, 1/\beta, 0)$ and $(0, 0, 1/\gamma)$ respectively.

The Cartesian coordinates of A are thus $(\lambda_1/\alpha, \lambda_2/\alpha)$, of B $(\mu_1/\beta, \mu_2/\beta)$, and of C $(\nu_1/\gamma, \nu_2/\gamma)$.

All Cartesian equations of algebraical curves transform into *homogeneous* equations in X, Y, Z by reason of the relation (3). The general equation of the second degree may thus be written

$$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY=0,$$

and this of course represents a conic. We shall later on discriminate the nature of the conic thus represented.

305. Area of triangle.

We proceed now to find an expression for the area of a triangle PQR the coordinates of whose vertices are $(X_1Y_1Z_1)$, $(X_2Y_2Z_2)$, $(X_3Y_3Z_3)$ respectively.

Let (x_1y_1), (x_2y_2), (x_3y_3) be the Cartesian coordinates of the vertices P, Q, R.

The algebraical area of the triangle is then

$$\tfrac{1}{2}\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}\sin\omega,$$

where ω is the angle between the axes. And this

$$=\tfrac{1}{2}\begin{vmatrix} \lambda_1X_1+\mu_1Y_1+\nu_1Z_1, & \lambda_1X_2+\mu_1Y_2+\nu_1Z_2, & \lambda_1X_3+\mu_1Y_3+\nu_1Z_3 \\ \lambda_2X_1+\mu_2Y_1+\nu_2Z_1, & \lambda_2X_2+\mu_2Y_2+\nu_2Z_2, & \lambda_2X_3+\mu_2Y_3+\nu_2Z_3 \\ \alpha X_1+\beta Y_1+\gamma Z_1, & \alpha X_2+\beta Y_2+\gamma Z_2, & \alpha X_3+\beta Y_3+\gamma Z_3 \end{vmatrix}\sin\omega$$

$$=\tfrac{1}{2}\begin{vmatrix} X_1, & X_2, & X_3 \\ Y_1, & Y_2, & Y_3 \\ Z_1, & Z_2, & Z_3 \end{vmatrix}\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \alpha, & \beta, & \gamma \end{vmatrix}\sin\omega.$$

And in particular the algebraical area of the triangle of reference ABC is

$$\frac{1}{2}\begin{vmatrix} \frac{1}{\alpha}, & 0, & 0 \\ 0, & \frac{1}{\beta}, & 0 \\ 0, & 0, & \frac{1}{\gamma} \end{vmatrix}\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \alpha, & \beta, & \gamma \end{vmatrix}\sin\omega$$

$$=\frac{1}{2}\cdot\frac{1}{\alpha\beta\gamma}\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \alpha, & \beta, & \gamma \end{vmatrix}\sin\omega.$$

Denoting this by Δ we have that the algebraical area of the triangle PQR is

$$\alpha\beta\gamma\Delta\begin{vmatrix} X_1, & X_2, & X_3 \\ Y_1, & Y_2, & Y_3 \\ Z_1, & Z_2, & Z_3 \end{vmatrix}.$$

306. Geometrical interpretation of the new coordinates.

The expression thus obtained for the area of a triangle gives us a geometrical meaning for our new coordinates.

For if (X, Y, Z) be the coordinates of any point P, the area of the triangle PBC is

$$\alpha\beta\gamma\Delta\begin{vmatrix} X, & Y, & Z \\ 0, & \frac{1}{\beta}, & 0 \\ 0, & 0, & \frac{1}{\gamma} \end{vmatrix}=\alpha\Delta X.$$

Thus $\alpha X=\frac{\Delta PBC}{\Delta ABC}$. Similarly $\beta Y=\frac{\Delta PCA}{\Delta BCA}$ and $\gamma Z=\frac{\Delta PAB}{\Delta CAB}$.

Thus we learn that

X is $1/\alpha$ times the ratio of the area of ΔPBC to that of ΔABC.
Y is $1/\beta$ „ „ „ ΔPCA „ „ „
Z is $1/\gamma$ „ „ „ ΔPAB „ „ „

We see then that areal coordinates are the special case where $\alpha = \beta = \gamma = 1$.

307. Transformation to Cartesians.

Now let $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ be the Cartesian coordinates of the vertices A, B, C of the triangle of reference. We therefore have from the relations

$$x = \lambda_1 X + \mu_1 Y + \nu_1 Z \qquad y = \lambda_2 X + \mu_2 Y + \nu_2 Z$$

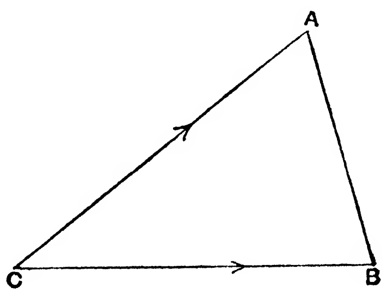
that $x_1 = \dfrac{\lambda_1}{\alpha}$, $y_1 = \dfrac{\lambda_2}{\alpha}$; $x_2 = \dfrac{\mu_1}{\beta}$, $y_2 = \dfrac{\mu_2}{\beta}$; $x_3 = \dfrac{\nu_1}{\gamma}$, $y_3 = \dfrac{\nu_2}{\gamma}$.

Therefore the general relations connecting together the Cartesian and the other coordinates can be written

$$x = \alpha x_1 X + \beta x_2 Y + \gamma x_3 Z, \qquad y = \alpha y_1 X + \beta y_2 Y + \gamma y_3 Z.$$

By means of these relations we can pass from our new coordinates to Cartesian coordinates with any axes if the vertices of the triangle of reference be known in relation to the Cartesian axes.

In particular if we take as our Cartesian axes the sides CB, CA of the triangle of reference, the coordinates of the vertices of the triangle are $(0, b)$, $(a, 0)$, $(0, 0)$.



We thus get $\qquad x = \beta a Y, \qquad y = \alpha b X,$

whence $\quad X = \dfrac{1}{\alpha} \cdot \dfrac{y}{b}, \quad Y = \dfrac{1}{\beta} \cdot \dfrac{x}{a}, \quad Z = \dfrac{1}{\gamma}\left(1 - \dfrac{x}{a} - \dfrac{y}{b}\right).$

308. By transformation to Cartesian axes CB, CA we can shew in exactly the same way as for areal coordinates (§§ 261, 262, 270) that

(i) The coordinates of a point dividing the line joining (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) in the ratio $k : l$ are

$$\frac{kX_2 + lX_1}{k+l}, \quad \frac{kY_2 + lY_1}{k+l}, \quad \frac{kZ_2 + lZ_1}{k+l}.$$

(ii) The square of the line joining (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) is

$$-a^2\beta\gamma(Y_1 - Y_2)(Z_1 - Z_2) - b^2\gamma\alpha(Z_1 - Z_2)(X_1 - X_2) - c^2\alpha\beta(X_1 - X_2)(Y_1 - Y_2).$$

(iii) The equations of a line through (X_1, Y_1, Z_1) can be expressed in the form

$$\frac{X - X_1}{l} = \frac{Y - Y_1}{m} = \frac{Z - Z_1}{n} = r,$$

where r is the algebraical distance of (X, Y, Z) from (X_1, Y_1, Z_1), and l, m, n are constants for the line such that

$$\alpha l + \beta m + \gamma n = 0,$$
$$a^2\beta\gamma mn + b^2\gamma\alpha nl + c^2\alpha\beta lm = -1.$$

The student is recommended to work out these results for himself. He will observe too that the equation of the line at infinity is now

$$\alpha X + \beta Y + \gamma Z = \underset{\epsilon=0}{\mathrm{Lt}}\, \epsilon(lX + mY + nZ),$$

which we may write $\alpha X + \beta Y + \gamma Z = 0$,
this being interpreted to mean that $\alpha X + \beta Y + \gamma Z$ is finite even when X, Y, Z are some or all of them infinite (see §§ 265, 266, 278, 286, 300 a).

309. General equation of the second degree.

We come now to the general equation of the second degree, viz.

$$f(X, Y, Z) \equiv AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY = 0.$$

We shall have exactly as before that the tangent at (X_1, Y_1, Z_1) is

$$T \equiv (AX_1 + HY_1 + GZ_1)X + (HX_1 + BF_1 + FZ_1)Y + (GX_1 + FY_1 + CZ_1)Z = 0,$$

the form for the chord of contact and the polar of (X_1, Y_1, Z_1), when it is not on the curve, being the same as this.

The condition that the line $lX + mY + nZ = 0$ should touch the conic will be as before

$$\begin{vmatrix} A, & H, & G, & l \\ H, & B, & F, & m \\ G, & F, & C, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0.$$

The equation of the pair of tangents from (X_1, Y_1, Z_1) will be

$$f(X, Y, Z)\, f(X_1, Y_1, Z_1) = T^2.$$

310. The centre and the asymptotes.

The centre being the pole of the line at infinity we have if (X_1, Y_1, Z_1) be its coordinates,

$$\frac{AX_1 + HY_1 + GZ_1}{\alpha} = \frac{HX_1 + BY_1 + FZ_1}{\beta}$$
$$= \frac{GX_1 + FY_1 + CZ_1}{\gamma} = \lambda \text{ (say)},$$

whence
$$AX_1 + HY_1 + GZ_1 - \alpha\lambda = 0,$$
$$HX_1 + BY_1 + FZ_1 - \beta\lambda = 0,$$
$$GX_1 + FY_1 + CZ_1 - \gamma\lambda = 0,$$
and
$$\alpha X_1 + \beta Y_1 + \gamma Z_1 - 1 = 0,$$
from which we get on eliminating X_1, Y_1, Z_1,

$$\begin{vmatrix} A, & H, & G, & \alpha\lambda \\ H, & B, & F, & \beta\lambda \\ G, & F, & C, & \gamma\lambda \\ \alpha, & \beta, & \gamma, & 1 \end{vmatrix} = 0,$$

from which we find that

$$\lambda \begin{vmatrix} A, & H, & G, & \alpha \\ H, & B, & F, & \beta \\ G, & F, & C, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} + \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0.$$

The equation of the asymptotes can be shewn as in § 290 to be

$$f(X, Y, Z)\begin{vmatrix} A, & H, & G, & \alpha \\ H, & B, & F, & \beta \\ G, & F, & C, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} + (\alpha X + \beta Y + \gamma Z)^2 \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = 0.$$

311. Discrimination of the nature of the conic.

If we transform to Cartesian axes CB, CA the terms of the highest order are easily seen to be

$$\frac{1}{a^2}\left(\frac{B}{\beta^2} + \frac{C}{\gamma^2} - \frac{2F}{\beta\gamma}\right)x^2 + \frac{2}{ab}\left(\frac{C}{\gamma^2} + \frac{H}{\alpha\beta} - \frac{F}{\beta\gamma} - \frac{G}{\gamma\alpha}\right)xy$$
$$+ \frac{1}{b^2}\left(\frac{C}{\gamma^2} + \frac{A}{\alpha^2} - \frac{2G}{\gamma\alpha}\right)y^2.$$

From these we can at once obtain as in § 274 the conditions for a circle, viz.

$$\frac{\dfrac{B}{\beta^2} + \dfrac{C}{\gamma^2} - \dfrac{2F}{\beta\gamma}}{a^2} = \frac{\dfrac{C}{\gamma^2} + \dfrac{A}{\alpha^2} - \dfrac{2G}{\gamma\alpha}}{b^2} = \frac{\dfrac{A}{\alpha^2} + \dfrac{B}{\beta^2} - \dfrac{2H}{\alpha\beta}}{c^2},$$

which we may write $\qquad \dfrac{p}{a^2} = \dfrac{q}{b^2} = \dfrac{r}{c^2},$

where p, q, r stand for the three numerators.

If the conic be not a circle it will be an ellipse, parabola or hyperbola according as

$$\left(\frac{C}{\gamma^2} + \frac{H}{\alpha\beta} - \frac{F}{\beta\gamma} - \frac{G}{\gamma\alpha}\right)^2 \lesseqqgtr \left(\frac{B}{\beta^2} + \frac{C}{\gamma^2} - \frac{2F}{\beta\gamma}\right)\left(\frac{C}{\gamma^2} + \frac{A}{\alpha^2} - \frac{2G}{\gamma\alpha}\right),$$

which we may easily get into the form

$$p^2 + q^2 + r^2 - 2pq - 2qr - 2rp \lesseqqgtr 0.$$

Or the condition can be expressed in determinantal form

$$\begin{vmatrix} \dfrac{A}{\alpha^2}, & \dfrac{H}{\alpha\beta}, & \dfrac{G}{\gamma\alpha}, & 1 \\ \dfrac{H}{\alpha\beta}, & \dfrac{B}{\beta^2}, & \dfrac{F}{\beta\gamma}, & 1 \\ \dfrac{G}{\gamma\alpha}, & \dfrac{F}{\beta\gamma}, & \dfrac{C}{\gamma^2}, & 1 \\ 1, & 1 & 1 & 0 \end{vmatrix} \lesseqqgtr 0,$$

which can be written

$$\frac{1}{\alpha^2\beta^2\gamma^2}\begin{vmatrix} A, & H, & G, & \alpha \\ H, & B, & F, & \beta \\ G, & F, & C, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} \lesseqgtr 0,$$

that is, for we are supposing α, β, γ all real,

$$\begin{vmatrix} A, & H, & G, & \alpha \\ H, & B, & F, & \beta \\ G, & F, & C, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} \lesseqgtr 0.$$

The condition for a rectangular hyperbola can easily be proved as in § 276 to be

$$\frac{p\cos A}{a}+\frac{q\cos B}{b}+\frac{r\cos C}{c}=0.$$

It must be clearly understood that this discrimination we have made is not applicable if the coordinates be imaginary. The need for this caution will be apparent at a later stage. The coordinates will be imaginary if some of the quantities λ, μ, ν, α, β, γ of § 304 be imaginary.

312. Special conics.

The student will be able to see for himself that the general equation of conics circumscribed to the triangle of reference, *whatever system of homogeneous coordinates is used*, is of the form

$$Fyz+Gzx+Hxy=0,$$

that conics inscribed in the triangle of reference have for their equation

$$\sqrt{\lambda x}+\sqrt{\mu y}+\sqrt{\nu z}=0,$$

and that $$Ax^2+By^2+Cz^2=0$$

is the equation of conics for which the triangle of reference is self-polar.

The equation of the circle circumscribing the triangle of reference is

$$\frac{a^2}{\alpha x}+\frac{b^2}{\beta y}+\frac{c^2}{\gamma z}=0,$$

and in general it will be observed that when the equation of any conic related to the triangle of reference is known in areal coordinates, its equation in generalised homogeneous coordinates, for which the line at infinity is $\alpha x + \beta y + \gamma z = 0$, will be got from the areal equation by writing αx, βy, γz for x, y, z respectively.

313. The foci.

To find the foci of the general conic in any system of homogeneous coordinates.

Let $S \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ be the conic.

The pair of tangents from (x_1, y_1, z_1) is

$$S_1(Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy) = (\xi_1 x + \eta_1 y + \zeta_1 z)^2,$$

where $\quad \xi_1 = Ax_1 + Hy_1 + Gz_1, \quad \eta_1 = Hx_1 + By_1 + Fz_1,$

$$\zeta_1 = Gx_1 + Fy_1 + Cz_1.$$

That this may satisfy the conditions for a circle we must have

$$\frac{\left(\frac{B}{\beta^2} + \frac{C}{\gamma^2} - \frac{2F}{\beta\gamma}\right)S_1 - \frac{\eta_1^2}{\beta^2} - \frac{\zeta_1^2}{\gamma^2} + \frac{2\eta_1\zeta_1}{\beta\gamma}}{a^2} = \text{similar expressions.}$$

That is

$$\frac{pS_1 - \left(\frac{\eta_1}{\beta} - \frac{\zeta_1}{\gamma}\right)^2}{a^2} = \frac{qS_1 - \left(\frac{\zeta_1}{\gamma} - \frac{\xi_1}{\alpha}\right)^2}{b^2} = \frac{rS_1 - \left(\frac{\xi_1}{\alpha} - \frac{\eta_1}{\beta}\right)^2}{c^2},$$

where $\quad p \equiv \frac{B}{\beta^2} + \frac{C}{\gamma^2} - \frac{2F}{\beta\gamma}$, $q \equiv$ etc., as before.

These equations then determine the foci.

314. The axes.

To obtain the equation of the axes of the general conic in any system of homogeneous coordinates.

We see from the last paragraph that the coordinates of the foci satisfy

$$\frac{pS - \left(\frac{\eta}{\beta} - \frac{\zeta}{\gamma}\right)^2}{a^2} = \frac{qS - \left(\frac{\zeta}{\gamma} - \frac{\xi}{\alpha}\right)^2}{b^2} = \frac{rS - \left(\frac{\xi}{\alpha} - \frac{\eta}{\beta}\right)^2}{c^2} = \lambda \text{ (say)},$$

where $\xi = Ax + Hy + Gz, \quad \eta = Hx + By + Fz,$
$$\zeta = Gx + Fy + Cz.$$

The above equations give

$$\left(\frac{\eta}{\beta} - \frac{\zeta}{\gamma}\right)^2 - pS + \lambda a^2 = 0,$$

$$\left(\frac{\zeta}{\gamma} - \frac{\xi}{\alpha}\right)^2 - qS + \lambda b^2 = 0,$$

$$\left(\frac{\xi}{\alpha} - \frac{\eta}{\beta}\right)^2 - rS + \lambda c^2 = 0.$$

Eliminating S and λ we have

$$\begin{vmatrix} \left(\frac{\eta}{\beta} - \frac{\zeta}{\gamma}\right)^2, & p, & a^2 \\ \left(\frac{\zeta}{\gamma} - \frac{\xi}{\alpha}\right)^2, & q, & b^2 \\ \left(\frac{\xi}{\alpha} - \frac{\eta}{\beta}\right)^2, & r, & c^2 \end{vmatrix} = 0 \quad \text{.............(A).}$$

This then is a conic on which the four foci lie. But this conic passes through the centre of the given conic, for the centre is given by

$$\frac{\xi}{\alpha} = \frac{\eta}{\beta} = \frac{\zeta}{\gamma}.$$

Hence (A) is the equation of the axes (comp. § 289).

The director circle.

It may be left as an exercise for the student to prove as in § 285 that the equation of the director circle is

$$\frac{\cos A}{a}\left\{pS - \left(\frac{\eta}{\beta} - \frac{\zeta}{\gamma}\right)^2\right\} + \frac{\cos B}{b}\left\{qS - \left(\frac{\zeta}{\gamma} - \frac{\xi}{\alpha}\right)^2\right\}$$
$$+ \frac{\cos C}{c}\left\{rS - \left(\frac{\xi}{\alpha} - \frac{\eta}{\beta}\right)^2\right\} = 0.$$

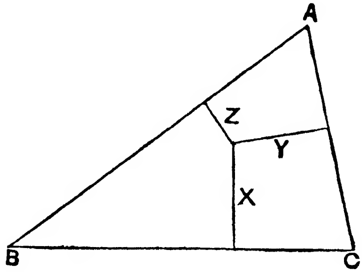
315. Trilinear coordinates.

Strictly speaking all homogeneous coordinates as we have defined them are trilinear in the sense that they are interpreted in reference to a triangle.

But the term 'trilinear coordinates' has been specially used for that system where the position of a point is determined by its perpendiculars upon the sides of the triangle of reference. Denoting these by X, Y, Z we have the identical relation,

$$aX + bY + cZ = 2\Delta,$$

where Δ is the area of the triangle.



So for this particular system we have

$$\frac{a}{2\Delta} X + \frac{b}{2\Delta} Y + \frac{c}{2\Delta} Z = 1.$$

The equation of the line at infinity is thus

$$aX + bY + cZ = 0.$$

The letters α, β, γ are frequently used instead of X, Y, Z in this system.

The student may practise himself by shewing that the condition that the general conic should be a rectangular hyperbola in this system is

$$A + B + C - 2F \cos\alpha - 2G \cos\beta - 2H \cos\gamma = 0,$$

where α, β, γ are the angles of the triangle of reference.

316. Cartesian coordinates as a homogeneous system.

Suppose we transform our Cartesian coordinates x, y by writing

$$x = X, \qquad y = Y, \qquad 1 = \alpha X + \beta Y + Z.$$

The sides of the triangle of reference being $X = 0$, $Y = 0$, $Z = 0$, are in the Cartesians

$$x = 0, \quad y = 0, \quad \text{and} \quad \alpha x + \beta y = 1,$$

that is to say, the axes of coordinates and the line $\alpha x + \beta y = 1$.

Now when α and β are very small this last line has very large intercepts on the axes, and in the limit when α and β approach zero it becomes the line at infinity, we then have

$$x = X, \quad y = Y, \quad 1 = Z,$$

and we have a homogeneous system in which the triangle of reference is formed by the axes of Cartesian coordinates and the line at infinity.

We may thus make all our Cartesian equations homogeneous by the insertion of appropriate powers of Z, which is unity.

We can thus include Cartesians among homogeneous coordinates and write the general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

z being unity.

It will be remembered that the equation of a central conic referred to two conjugate diameters as Cartesian axes was found to be

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

where α and β are the lengths of the semidiameters.

Making this homogeneous by means of z we have

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - z^2 = 0.$$

Thus we see that the equation of a conic referred to two conjugate diameters is really only a particular case of a conic for which the triangle of reference is a self-polar triangle. In this case the triangle of reference is formed by the two conjugate diameters and the line at infinity, which is of course the polar of the centre.

317. Transformation from one set of homogeneous coordinates to another.

We can transform from one set of homogeneous coordinates X, Y, Z to other coordinates X', Y', Z' by substitutions, such as

$$X = l_1X' + m_1Y' + n_1Z',$$
$$Y = l_2X' + m_2Y' + n_2Z',$$
$$Z = l_3X' + m_3Y' + n_3Z',$$

then X', Y', Z' form a homogeneous system, and as

$$\alpha X + \beta Y + \gamma Z = 1,$$

there is a relation between X', Y', Z', such as

$$\alpha' X' + \beta' Y' + \gamma' Z' = 1.$$

We see now that by a proper transformation we can reduce the equation of a conic to a simple form. For in any system of homogeneous coordinates the equation of a conic for which the triangle of reference is self-conjugate is

$$AX^2 + BY^2 + CZ^2 = 0.$$

Thus by taking such a triangle for the triangle of reference we can express the equation of a conic in this form, and we can then write

$$X\sqrt{A} = x, \qquad Y\sqrt{B} = y, \qquad Z\sqrt{C} = z,$$

and we have a new set of homogeneous coordinates x, y, z in which the equation of the conic is

$$x^2 + y^2 + z^2 = 0.$$

Clearly x, y, z cannot be all real here if the conic be real, and the test for discriminating the conic cannot be applied.

318. Polar reciprocals.

Let us make use of homogeneous coordinates to prove that the polar reciprocal of a conic S with respect to a conic Γ is a conic.

The polar reciprocal of S with respect to Γ means the locus of the poles of tangents to S with respect to Γ.

We may take as the equation of Γ,

$$x^2 + y^2 + z^2 = 0 \quad\text{..........................(1)},$$

and represent S by the general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad\text{.........(2)}.$$

Let (x_1, y_1, z_1) be the pole with respect to (1) of some tangent to (2).

The equation of the polar being

$$xx_1 + yy_1 + zz_1 = 0,$$

we see that this must satisfy the condition for being a tangent to (2),

$$\therefore \begin{vmatrix} a, & h, & g, & x_1 \\ h, & b, & f, & y_1 \\ g, & f, & c, & z_1 \\ x_1, & y_1, & z_1, & 0 \end{vmatrix} = 0.$$

Thus the polar reciprocal of S with respect to Γ is

$$\begin{vmatrix} a, & h, & g, & x \\ h, & b, & f, & y \\ g, & f, & c, & z \\ x, & y, & z, & 0 \end{vmatrix} = 0,$$

which we may write

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{......(3)},$$

where A, B, C, etc. are the minors, with their proper signs, of a, b, c, etc. in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \equiv \Delta.$$

Thus the polar reciprocal of S with respect to Γ is the conic (3) which we will call S'.

319. It is now easy to see that if we take the polar reciprocal of S' with respect to Γ we get the conic S.

For the polar reciprocal will be

$$A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0 \quad \text{...(4)},$$

where A', B', etc. are the minors of A, B, etc. in the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}.$$

$$\therefore A' = (BC - F^2) = (ca - g^2)(ab - h^2) - (gh - af)^2 = a\Delta.$$

Similarly $B' = b\Delta$, $C' = c\Delta$, $F' = f\Delta$,

and so on, so that the conic (4) is the same as

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

320. The following special cases should be verified:

The polar reciprocal with respect to $x^2 + y^2 + z^2 = 0$ of

(i) $Ax^2 + By^2 + Cz^2 = 0$ is $\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0$,

(ii) $Fyz + Gzx + Hxy = 0$ is $\sqrt{Fx} + \sqrt{Gy} + \sqrt{Hz} = 0$.

321. Conics expressed in terms of a single parameter.

The homogeneous coordinates (x, y, z) of any point on a conic are in an infinite number of ways given by the ratios

$$x : y : z = at^2 + bt + c : a't^2 + b't + c' : a''t^2 + b''t + c'',$$

where the a's, b's, and c's are all constants, and t is a variable parameter.

For if we have a pair of tangents of the conic whose equations are

$$l_1x + m_1y + n_1z = 0, \qquad l_2x + m_2y + n_2z = 0,$$

and if the equation of their chord of contact be

$$l_3x + m_3y + n_3z = 0,$$

the equation of the conic will be of the form

$$(l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z) = \lambda (l_3x + m_3y + n_3z)^2,$$
$$= (lx + my + nz)^2 \text{ (say).}$$

Thus $$\frac{l_1x + m_1y + n_1z}{lx + my + nz} = \frac{lx + my + nz}{l_2x + m_2y + n_2z}.$$

Putting each of these ratios equal to t we have

$$(l_1 - lt)x + (m_1 - mt)y + (n_1 - nt)z = 0,$$
$$(l - l_2t)x + (m - m_2t)y + (n - n_2t)z = 0.$$

Whence

$$\frac{x}{at^2 + bt + c} = \frac{y}{a't^2 + b't + c'} = \frac{z}{a''t^2 + b''t + c''},$$

where a, a', etc. are constants, being functions of l, m, etc.

As we can express the conic in an infinite number of ways by taking different pairs of tangents and their chord of contact, we see that this mode of representing the points on the conic can be effected in an infinite number of ways.

322. On the equations of two conics.

We have seen that the equation of a single conic can be written in the simple form $x^2+y^2+z^2=0$. The question we have now to consider is, how best to choose our coordinates so as to have the equations of *two* conics in as simple a form as possible.

Now we know that if P, Q, R, S be four distinct points on a conic and if PQ and RS meet in A, PR and QS meet in B, and PS and QR in C, the triangle ABC is self-polar for the conic (*Course of Pure Geometry*, § 119a).

Now two conics in general cut in four points. We see then that *if the four points of intersection of two conics be distinct*, that is if no two of them coincide, the conics have a common self-polar triangle.

Thus the equations of the two conics can, if we take this self-polar triangle for the triangle of reference, be expressed in the form

$$ax^2+by^2+cz^2=0, \qquad a'x^2+b'y^2+c'z^2=0.$$

We may now write

$$x\sqrt{a}=X, \quad y\sqrt{b}=Y, \quad z\sqrt{c}=Z$$

and so get the equations of the two conics in the form

$$X^2+Y^2+Z^2=0, \qquad AX^2+BY^2+CZ^2=0.$$

323. The common self-polar triangle.

It must not however be supposed that the common self-polar triangle of two conics intersecting in four distinct points is always real. It is clearly real if the four points of intersection be real; we shall shew that it is also real if the four points of intersection be all imaginary, but that if two of the four points be real and two imaginary the self-polar triangle has one real and two imaginary vertices.

For taking the case where the four points of intersection P, Q, R, S are all imaginary, their Cartesian coordinates referred to any axes will be of the form

$$(\alpha_1+i\beta_1, \gamma_1+i\delta_1), (\alpha_1-i\beta_1, \gamma_1-i\delta_1), (\alpha_2+i\beta_2, \gamma_2+i\delta_2),$$
$$(\alpha_2-i\beta_2, \gamma_2-i\delta_2),$$

since the imaginary points of intersection of two real conics must occur in pairs. It will easily be seen that the lines PQ and RS are real and so their intersection is real. The equation of the line PR will be of the form

$$Ax + By + C + i(A'x + B'y + C') = 0$$

where A, B, C, A', etc. are all real.

The equations of QS will be obtained from this by writing $-i$ for i, that is the equation of QS will be

$$Ax + By + C - i(A'x + B'y + C') = 0.$$

Thus the point of intersection of PR and QS will be that of the lines

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0,$$

which is real.

Similarly the intersection of PS and QR is real.

Thus the self-polar triangle is real.

But if the conics have two real points of intersection $P(\alpha_1, \gamma_1)$, $Q(\alpha_2, \gamma_2)$ and two imaginary points $R(\alpha_3 + i\beta_3, \gamma_3 + i\delta_3)$ and $S(\alpha_3 - i\beta_3, \gamma_3 - i\delta_3)$. It will be seen that PQ and RS are both real and therefore their intersection is real. But the intersections of PR and QS and of PS and QR are both imaginary, for an imaginary line (and PR, PS are such) cannot contain more than one real point; for the line through two real points is real.

Thus the common self-polar triangle is imaginary.

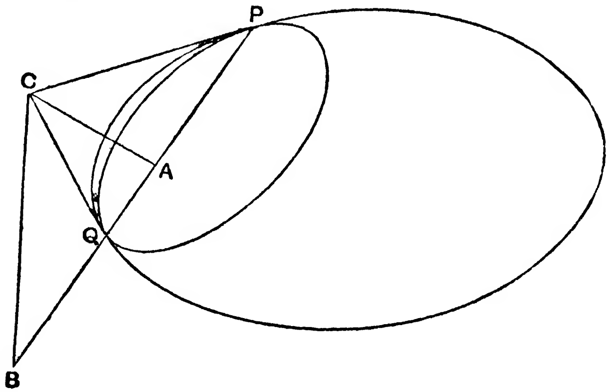
324. Double contact.

We now proceed to consider the cases where the four points of intersection of the two conics which we will denote by S and S' are not all distinct*.

Consider first the case where the conics have double contact; that is touch at two points. We consider this case first because the conics have a common self-polar triangle. Let the conics touch at P and Q. Let their common tangents at P and Q meet in C.

* I have made use of the Cambridge Tract on *Quadratic Forms* by T. J. I'A. Bromwich, Sc.D.

Take any two points A and B on PQ which are conjugate points with regard to S.



Take ABC for the triangle of reference.

Thus the equation of S will be of the form

$$ax^2 + by^2 + cz^2 = 0.$$

Now S' is a conic having double contact with S at the points where it is cut by $z = 0$; thus its equation will be of the form

$$ax^2 + by^2 + cz^2 = \lambda z^2,$$

which we may write

$$ax^2 + by^2 + c'z^2 = 0.$$

These then can be reduced to

$$X^2 + Y^2 + Z^2 = 0, \qquad X^2 + Y^2 + CZ^2 = 0.$$

We see then that ABC is self-polar for S' as well as for S, and as we have an infinite number of possible positions for A and B on the line PQ the two conics have an infinite number of self-polar triangles.

Moreover we have an infinite number of *real* self-polar triangles. This is clearly the case if P and Q are real. If P and Q are imaginary their Cartesian coordinates can be expressed $(\alpha + i\beta, \gamma + i\delta)$, $(\alpha - i\beta, \gamma - i\delta)$. The tangents at these points will intersect in a real point. Therefore C is real, and moreover the line PQ is real and so an infinite number of pairs of real conjugate points can be taken.

325. Single contact.

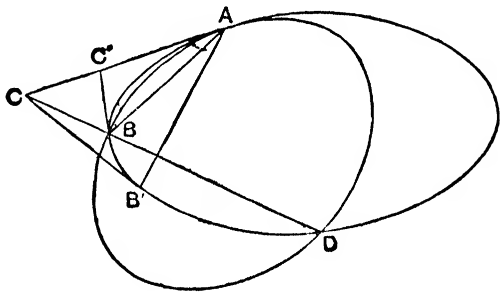
Let the conics S and S' touch at A and cut in B and D. Draw the common tangent at A and let the chord BD meet this in C.

From C draw CB' tangent to S.

Take ACB' for triangle of reference.

The equation of S will thus be of the form $2xy + \lambda z^2 = 0$.

Also the common chords of S' with S are $y = 0$ and a line through C whose equation will be of the form $kx + ly = 0$.



Thus the equation of S' will take the form

$$2xy + \lambda z^2 + y(kx + ly) = 0,$$

that is the form $\qquad by^2 + \lambda z^2 + 2hxy = 0.$

Writing $z\sqrt{\lambda} = Z$ we see that two conics with single contact can be expressed in the form

$$z^2 + 2xy = 0,$$
$$by^2 + z^2 + 2hxy = 0.$$

Or again if we draw the tangent to S at B to meet the tangent at A in C' and take $C'AB$ for triangle of reference, the equation of S will take the form $z^2 + 2xy = 0$, and S' will be of the form $z^2 + 2xy + y(kx + lz) = 0$, that is $z^2 + 2fyz + 2hxy = 0$.

Again we might take ABD for triangle of reference, in which case the conics would be of the form

$$fyz + gzx + hxy = 0, \quad f'yz + g'zx + h'xy = 0.$$

The tangents at A being

$$gz + hy = 0, \quad g'z + h'y = 0,$$

and being the same line, we have

$$\frac{g'}{g} = \frac{h'}{h} = \lambda \text{ (say).}$$

∴ the conics will be

$$fyz + gzx + hxy = 0, \quad f'yz + \lambda(gzx + hxy) = 0.$$

There is no common self-conjugate triangle in this case.

326. Three point contact.

Let the conics have three point contact at A and cut in B.

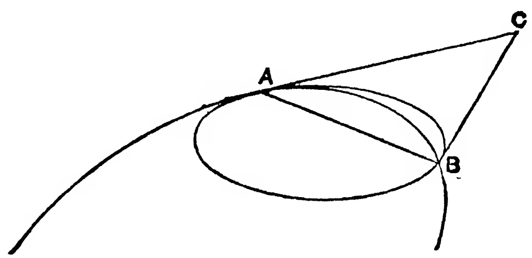
Let the tangent to S at B cut the common tangent at A in C.

Take ABC for the triangle of reference.

The equation of S is of form $2xy = \lambda z^2$.

And S' has with S the pair of common chords $y = 0$, $z = 0$.

Hence its equation is of the form $2xy - \lambda z^2 = \mu yz$.



In this case then the equations of the conics can be written

$$2hxy + cz^2 = 0, \qquad 2f'yz + 2hxy + cz^2 = 0,$$

which again might be reduced as before to

$$2XY + Z^2 = 0, \qquad 2FYZ + 2XY + Z^2 = 0.$$

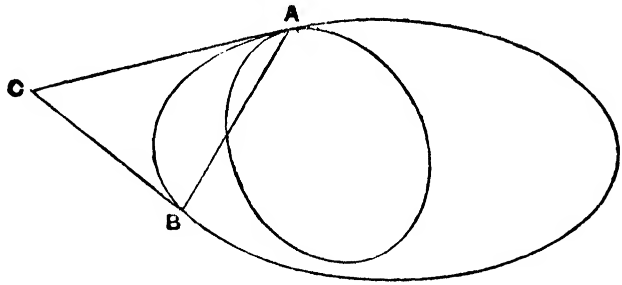
There is no common self-conjugate triangle.

327. Four point contact.

Let the conics S and S' have four point contact at A.

Let C be any point on the tangent at A. Draw CB the other tangent to S, and take ABC for the triangle of reference.

The equation of S is of the form $2xy=\lambda z^2$, and the pair of common chords of S' with S being $y=0$ twice over, the equation of S' will be $2xy-\lambda z^2=\mu y^2$.



Thus the equations this time are of the form

$$2hxy+cz^2=0, \qquad b'y^2+2hxy+cz^2=0,$$

which we can reduce to

$$2XY+Z^2=0, \qquad BY^2+2XY+Z^2=0.$$

Here again there is no common self-conjugate triangle.

EXAMPLES.

1. If $\alpha x+\beta y+\gamma z=0$ be the equation of the line at infinity in any system of homogeneous coordinates and a, b, c the lengths of the sides and A, B, C the angles of the triangle of reference, prove that the condition that the two lines

$$lx+my+nz=0, \quad l'x+m'y+n'z=0$$

should be at right angles to one another is

$$\frac{a^2}{\alpha^2}ll'+\frac{b^2}{\beta^2}mm'+\frac{c^2}{\gamma^2}nn'-\frac{bc}{\beta\gamma}(mn'+m'n)\cos A$$
$$-\frac{ca}{\gamma\alpha}(nl'+n'l)\cos B-\frac{ab}{\alpha\beta}(lm'+l'm)\cos C=0.$$

Simplify this in the special case of 'trilinear coordinates.'

2. Prove that the necessary and sufficient condition that the pair of lines

$$lx + my + nz = 0, \quad l'x + m'y + n'z = 0$$

should be conjugate lines for the general conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

in any homogeneous system is

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l', & m', & n', & 0 \end{vmatrix} = 0.$$

3. If the lines

$$l_1x + m_1y + n_1z = 0, \quad l_2x + m_2y + n_2z = 0, \quad l_3x + m_3y + n_3z = 0$$

form a triangle self-polar with respect to a conic for which the triangle of reference is self-polar, then

$$\begin{vmatrix} \frac{1}{l_1}, & \frac{1}{m_1}, & \frac{1}{n_1} \\ \frac{1}{l_2}, & \frac{1}{m_2}, & \frac{1}{n_2} \\ \frac{1}{l_3}, & \frac{1}{m_3}, & \frac{1}{n_3} \end{vmatrix} = 0.$$

4. The line $lx + my + nz = 0$ meets in E and F the sides AC, AB of the triangle of reference for any system of homogeneous coordinates. The equation of the line at infinity being known, obtain the equation of the line joining A to the middle point of EF.

5. In any system of homogeneous coordinates if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the vertices of a triangle inscribed in the conic

$$\frac{l}{x} + \frac{m}{y} + \frac{n}{z} = 0,$$

the sides of the triangle will touch the conic

$$l\sqrt{\frac{x}{x_1x_2x_3}} + m\sqrt{\frac{y}{y_1y_2y_3}} + n\sqrt{\frac{z}{z_1z_2z_3}} = 0.$$

Shew also that the triangle will be self-polar for the conic

$$\frac{lx^2}{x_1x_2x_3} + \frac{my^2}{y_1y_2y_3} + \frac{nz^2}{z_1z_2z_3} = 0.$$

6. Shew that if $abc = fgh$, any one of the three conics

$$ax^2 + 2fyz = 0, \quad by^2 + 2gzx = 0, \quad cz^2 + 2hxy = 0$$

is the polar reciprocal of a second with regard to the third.

7. Prove that the four points of intersection of the conic

$$yz + zx + xy = 0$$

with the conic

$$a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 - 2bcyz - 2cazx - 2abxy = 0$$

are concyclic, and find the equation of the circle through the four points, the coordinates being trilinear.

Transpose this problem into one with the notation of areal coordinates.

8. Two conics pass through the angular points of a given equilateral triangle and cut each other at right angles at these points; shew that the locus of their remaining intersection is

$$2x^2y^2z^2 - xyz(yz + zx + xy)(x + y + z) + (yz + zx + xy)^3,$$

the coordinates being trilinear and the equilateral triangle the triangle of reference.

State the corresponding equation if the coordinates were areal.

9. Prove that the locus of the centre of a conic which circumscribes the triangle of reference and touches the line

$$lx + my + nz = 0$$

is $\sqrt{lx(-ax + by + cz)} + \sqrt{my(ax - by + cz)} + \sqrt{nz(ax + by - cz)} = 0,$

a, b, c being the sides of the triangle of reference, the coordinates being 'trilinear.'

Give the corresponding equation when the coordinates are (i) areal, (ii) homogeneous coordinates for which the line at infinity is

$$\alpha x + \beta y + \gamma z = 0.$$

10. A conic is inscribed in a triangle and the trilinear coordinates of a focus with respect to this triangle are (x', y', z'); prove that the line

$$xx'(y'^2 - z'^2) + yy'(z'^2 - x'^2) + zz'(x'^2 - y'^2) = 0$$

is an axis of the conic.

11. Prove that in trilinear coordinates

$$a\sqrt{x\sin(B-C)}+b\sqrt{y\sin(C-A)}+c\sqrt{z\sin(A-B)}=0$$

is a parabola touching the sides of the triangle of reference, and having for its directrix the line through the centre of gravity and the orthocentre of the triangle.

Generalise this for any homogeneous coordinates.

12. Shew that the equation in trilinear coordinates (x, y, z) of a conic circumscribing the triangle of reference is

$$\frac{a}{px}+\frac{b}{qy}+\frac{c}{rz}=0,$$

where a, b, c are the sides of the triangle, and p, q, r are the focal chords parallel to these sides.

13. The coordinates being trilinear and referred to the triangle ABC, the equation of the directrix of the parabola

$$\lambda yz+\mu zx+\nu xy=0$$

is $$(\mu^2+\nu^2-2\mu\nu\cos A)\,bcx+(\nu^2+\lambda^2-2\nu\lambda\cos B)\,cay$$
$$+(\lambda^2+\mu^2-2\lambda\mu\cos C)\,abz=0.$$

14. Shew that two triangles whose sides pass through A, B, C respectively (the vertices of the triangle of reference) can be inscribed in the conic

$$\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0,$$

and that the equations of the lines joining corresponding vertices of the triangles are

$$lx+my-3nz=0, \text{ etc.}$$

15. S is a conic inscribed in a given triangle OMN, S' is a conic touching OM, ON at M and N and intersecting S in P and Q. Prove that tangents to S at P and Q will intersect on S'.

16. Prove that if two conics have four point contact at O, and Q be the pole with respect to the second of the tangent at P on the first O, P, Q are collinear.

17. The polar reciprocal of $y^2 - 4cx = 0$ with respect to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } a^2cy^2 + b^4x = 0.$$

Also the polar reciprocal of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to $y^2 - 4cx = 0$ is

$$b^2y^2 = 4c^2(x^2 - a^2).$$

18. A conic having (x_1, y_1) (x_2, y_2) for foci is reciprocated with respect to the circle $x^2 + y^2 = c^2$, shew that the equation of the reciprocal is of the form

$$k(x^2 + y^2) + (xx_1 + yy_1 - c^2)(xx_2 + yy_2 - c^2) = 0.$$

If the given conic pass through the origin determine k and shew that the latera recta of the two possible reciprocals are

$$c^2\left(\frac{1}{r_1} \sim \frac{1}{r_2}\right)\sin\tfrac{1}{2}(\theta_1 \sim \theta_2), \quad c^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\cos(\theta_1 \sim \theta_2),$$

where (r_1, θ_1) and (r_2, θ_2) are the polar coordinates of the given foci.

19. Shew that the polar reciprocal of $x^2 + y^2 - 2xx_0 - 2yy_0 = 0$ with reference to $xy - 1 = 0$ is given either by

$$(x_0^2 + y_0^2)(x^2 + y^2) - (xy_0 + yx_0 - 2)^2 = 0,$$

or
$$\sqrt{(y_0 - ix_0)(x + iy)} + \sqrt{(y_0 + ix_0)(x - iy)} + 2 = 0$$

and account for both of these forms.

20. If the reciprocal of one parabola with respect to another be a parabola, the three curves have their axes parallel or coincident.

21. The conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is reciprocated with respect to a point. Shew that if the reciprocal be always similar to the original curve the point must lie on the curve

$$a^2b^2(x^2 + y^2)^2 = (a^4 - b^4)(b^2x^2 - a^2y^2),$$

whereas if its area be constant, the point must lie on a conic similar to the original one.

22. The triangle formed by the polars of the middle points of the sides of a given triangle with respect to any inscribed conic is of constant area.

[The conic may be taken to be $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$.]

CHAPTER XVI.

CROSS RATIOS, HARMONIC SECTION, INVOLUTION.

328. Analytical representation of cross ratios.

It is assumed, as is explained in the preface to this work, that the reader is already familiar with the principles of cross ratios, harmonic section and involution. We are now concerned with the bearing of analytical geometry on this branch of the subject.

PROP. *The cross ratio of the pencil formed by the lines whose Cartesian equations are*

$$y = m_1 x, \quad y = m_2 x, \quad y = m_3 x, \quad y = m_4 x$$

is
$$\frac{(m_1 - m_2)(m_3 - m_4)}{(m_1 - m_4)(m_3 - m_2)}.$$

Draw a line parallel to the y-axis to cut the four lines, taken in order, in P, Q, R, S. Let this line cut the x-axis in L, and let $OL = x$. Then

$$LP = m_1 x, \quad LQ = m_2 x, \quad LR = m_3 x, \quad RS = m_4 x.$$

The cross ratio required $= (PQRS)$

$$= \frac{PQ \,.\, RS}{PS \,.\, RQ} = \frac{(LQ - LP)(LS - LR)}{(LS - LP)(LQ - LR)}$$

$$= \frac{(m_2 x - m_1 x)(m_4 x - m_3 x)}{(m_4 x - m_1 x)(m_2 x - m_3 x)} = \frac{(m_1 - m_2)(m_3 - m_4)}{(m_1 - m_4)(m_3 - m_2)}.$$

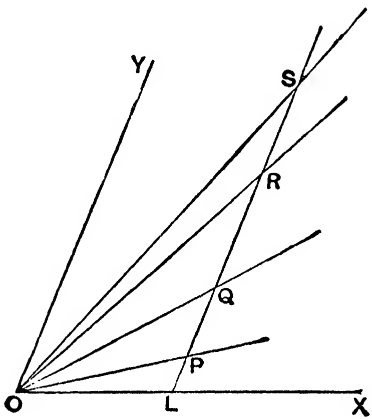
It is easy to see that the cross ratio of the pencil formed by the lines

$$x = m_1 y, \quad x = m_2 y, \quad x = m_3 y, \quad x = m_4 y$$

has this same value. This could be proved by drawing a line parallel to the x-axis. It is clear too from the fact that

$$\frac{(m_1 - m_2)(m_3 - m_4)}{(m_1 - m_4)(m_3 - m_2)}$$

is unchanged when we write for m_1, m_2, m_3, m_4 their reciprocals.



COR. The cross ratio of the pencil formed by the lines

$$X = \lambda_1 Y, \quad X = \lambda_2 Y, \quad X = \lambda_3 Y, \quad X = \lambda_4 Y,$$

drawn through the vertex C of the triangle of reference in any system of homogeneous coordinates is

$$\frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)}.$$

For if we transform to Cartesian coordinates (x, y) with CB and CA as axes we have

$$X = \frac{1}{\alpha}\frac{y}{b} = py \text{ (say)}, \qquad Y = \frac{1}{\beta}\frac{x}{a} = qx \text{ (say)},$$

so that the Cartesian equations of the lines are

$$y = \frac{q}{p}\lambda_1 x, \quad y = \frac{q}{p}\lambda_2 x, \quad y = \frac{q}{p}\lambda_3 x, \quad y = \frac{q}{p}\lambda_4 x.$$

Whence, from the proposition, the cross ratio of the pencil is

$$\frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)}.$$

Example. The cross ratio of the pencil formed by lines through a point parallel to the four lines whose equations in any homogeneous system are

$$l_1x+m_1y+n_1z=0,$$
$$l_2x+m_2y+n_2z=0,$$
$$l_3x+m_3y+n_3z=0,$$
$$l_4x+m_4y+n_4z=0,$$

is

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ \alpha, & \beta, & \gamma \end{vmatrix} \begin{vmatrix} l_3, & m_3, & n_3 \\ l_4, & m_4, & n_4 \\ \alpha, & \beta, & \gamma \end{vmatrix} \div \begin{vmatrix} l_1, & m_1, & n_1 \\ l_4, & m_4, & n_4 \\ \alpha, & \beta, & \gamma \end{vmatrix} \begin{vmatrix} l_3, & m_3, & n_3 \\ l_2, & m_2, & n_2 \\ \alpha, & \beta, & \gamma \end{vmatrix},$$

the line at infinity being $\alpha x+\beta y+\gamma z=0$.

329. Pairs of lines harmonically conjugate.

PROP. *The necessary and sufficient condition that the pair of lines whose Cartesian equations are*

$$y=m_1x, \quad y=m_2x$$

should be harmonically conjugate with the lines

$$y=m_1'x, \quad y=m_2'x$$

is
$$(m_1+m_2)(m_1'+m_2')=2(m_1m_2+m_1'm_2').$$

For the lines being harmonically conjugate the cross ratio of them in the order

$$y=m_1x, \quad y=m_1'x, \quad y=m_2x, \quad y=m_2'x$$

must be -1,

$$\therefore\ (m_1-m_1')(m_2-m_2')=-(m_1-m_2')(m_2-m_1'),$$

and this reduces to

$$(m_1+m_2)(m_1'+m_2')=2(m_1m_2+m_1'm_2').$$

This condition then is necessary and it can be shewn to be sufficient by working the algebra backwards.

COR. 1. *The necessary and sufficient conditions that the lines*

$$X=\lambda_1Y, \quad X=\lambda_2Y,$$

in any homogeneous system, should be harmonically conjugate with the lines

$$X=\lambda_1'Y, \quad X=\lambda_2'Y$$

is
$$(\lambda_1+\lambda_2)(\lambda_1'+\lambda_2')=2(\lambda_1\lambda_2+\lambda_1'\lambda_2').$$

This condition holds equally well too if the lines be

$$Y=\lambda_1 X,\quad Y=\lambda_2 X,\quad Y=\lambda_1' X,\quad Y=\lambda_2' X.$$

COR. 2. *The necessary and sufficient condition that the pair of lines $ax^2+2hxy+by^2=0$ should be harmonically conjugate with $a'x^2+2h'xy+b'y^2=0$ in Cartesians or any other system of homogeneous coordinates is $ab'+a'b=2hh'$.*

For let $\quad ax^2+2hxy+by^2\equiv b(y-m_1x)(y-m_2x)$,

so that $\quad m_1+m_2=-\dfrac{2h}{b}$, and $m_1m_2=\dfrac{a}{b}$,

and let $\quad a'x^2+2h'xy+b'y^2\equiv b'(y-m_1'x)(y-m_2'x)$,

so that $\quad m_1'+m_2'=-\dfrac{2h'}{b'}$, and $m_1'm_2'=\dfrac{a'}{b'}$.

The condition that the first pair should be harmonically conjugate with the second pair is thus

$$2\left(\frac{a}{b}+\frac{a'}{b'}\right)=\frac{4hh'}{bb'},$$

that is

$$ab'+a'b=2hh'.$$

Example. The necessary and sufficient condition that the two pairs of points on the x-axis given by $ax^2+2hx+b=0$ should be harmonically conjugate with the pair given by $a'x^2+2h'x+b'=0$ is $ab'+a'b=2hh'$.

330. *If the pair of lines $y=mx$, $y=m'x$ be harmonically conjugate with the axes of coordinates in Cartesians then $m'=-m$.*

This can be deduced from § 329 by considering the pairs of lines $y^2-(m+m')xy+mm'x^2=0$, $xy=0$ as harmonically conjugate. Or we may proceed as follows:

Let a line parallel to the y-axis cut the given lines in P and Q and the axes of coordinates in R and S, the last point being at infinity.

The condition for the harmonic relation is

$$(PQ, RS)=-1,$$

$$\therefore\ (PQRS)=-1,$$

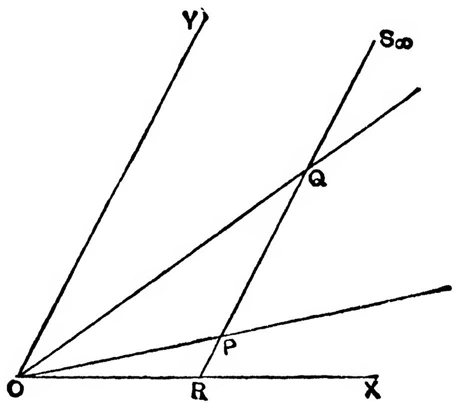
that is

$$\frac{PR\,.\,QS}{PS\,.\,QR}=-1,$$

$$\therefore\ PR = -QR,$$

for $\dfrac{QS}{PS} = 1$, S being at infinity,

$$\therefore\ RP = -RQ, \qquad \therefore\ m = -m'.$$



COR. *The pair of lines $x = \lambda y$, $x = -\lambda y$ in any homogeneous system is harmonically conjugate with the sides $x = 0$, $y = 0$ of the triangle of reference.*

This is at once seen by transforming to Cartesian axes CB, CA.

331. On the representation of four points in a plane.

If the triangle of reference be properly chosen four points in a plane (no three of which are collinear) can be represented in homogeneous coordinates by

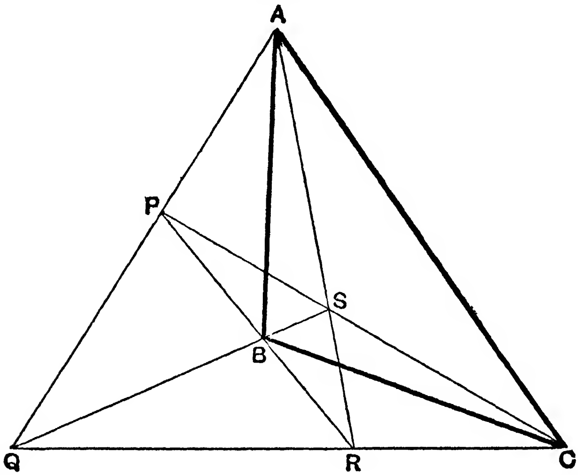
$$(f, g, h),\ (-f, g, h),\ (f, -g, h),\ (f, g, -h).$$

Let P, Q, R, S be the four points. Let ABC be the diagonal points of the quadrangle $PQRS$ (*Pure Geometry*, § 76). Take ABC for the triangle of reference. Let (f, g, h) be the coordinates of P. Then the equation of CP is

$$\frac{x}{f} = \frac{y}{g}.$$

But CQ and CP are harmonically conjugate with CA and CB. Therefore the equation of CQ is

$$\frac{x}{f} = -\frac{y}{g}. \qquad \text{(§ 330 Cor.)}$$



Also the equation of AP is

$$\frac{y}{g} = \frac{z}{h}.$$

Therefore, the point Q, being the intersection of AP and CQ, is given by

$$\frac{x}{-f} = \frac{y}{g} = \frac{z}{h},$$

that is $\qquad x_Q : y_Q : z_Q = -f : g : h.$

Similarly $\qquad x_R : y_R : z_R = f : -g : h,$

and $\qquad x_S : y_S : z_S = f : g : -h.$

Thus the four points can be represented by

$$(f, g, h),\ (-f, g, h),\ (f, -g, h),\ (f, g, -h).$$

But it must be noticed that if (f, g, h) be the actual coordinates of P, then $(-f, g, h)$ will not be the actual co-

ordinates of Q but only proportional to them. The actual coordinates of Q will be $(-\lambda f, \lambda g, \lambda h)$ where

$$\lambda(-\alpha f + \beta g + \gamma h) = 1.$$

A similar remark applies to the coordinates of R and S.

332. Conics through four points.

By a proper choice of the triangle of reference all conics through four given points (no three collinear) have their equation of the form

$$Ax^2 + By^2 + Cz^2 = 0,$$

where A, B, C are connected by a linear relation.

For we choose the triangle of reference as in the last paragraph and the coordinates of the four points are then represented by

$$(f, g, h),\ (-f, g, h),\ (f, -g, h),\ (f, g, -h).$$

Now take the general conic

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

and make it pass through these four points. We then get

$$F = G = H = 0$$

and $$Af^2 + Bg^2 + Ch^2 = 0.$$

Thus the equations of all conics through the four points are included in

$$Ax^2 + By^2 + Cz^2 = 0,$$

where the constants A, B, C which are different for the different conics are connected by the linear relation

$$Af^2 + Bg^2 + Ch^2 = 0.$$

We see then that the conics through P, Q, R, S all have the diagonal triangle ABC, which is the triangle of reference, for a self-polar triangle.

This can be proved otherwise from the harmonic properties of the quadrangle and of the pole and polar of a conic (*Pure Geometry*, § 119a).

333. PROP. *The locus of the centres of all conics through four given points is a conic circumscribing the diagonal triangle of the quadrangle.*

For taking as the typical equation of one of the conics,

$$Ax^2 + By^2 + Cz^2 = 0,$$

with the relation

$$Af^2 + Bg^2 + Ch^2 = 0,$$

we see that the coordinates of the centre are given by

$$\frac{Ax}{\alpha} = \frac{By}{\beta} = \frac{Cz}{\gamma} = \lambda \text{ (say)}, \qquad (\S\ 310)$$

$$\therefore\ A = \frac{\lambda\alpha}{x}, \quad B = \frac{\lambda\beta}{y}, \quad C = \frac{\lambda\gamma}{z}.$$

Thus the centre satisfies

$$\frac{\alpha f^2}{x} + \frac{\beta g^2}{y} + \frac{\gamma h^2}{z} = 0,$$

that is to say, the locus of the centre is a conic circumscribing the triangle of reference which is the diagonal triangle (compare *Pure Geometry*, § 218).

334. Representation of four lines in a plane.

By a proper choice of the triangle of reference the equations of four lines in a plane, no three of which are concurrent, can be expressed in the form

$$lx + my + nz = 0, \qquad -lx + my + nz = 0,$$
$$lx - my + nz = 0, \qquad lx + my - nz = 0.$$

Let the four lines form the quadrilateral $PQRS$. Let ABC, the triangle formed by its three diagonals PR, QS, DE, be taken as the triangle of reference. Let the equation of PQ be $lx + my + nz = 0$.

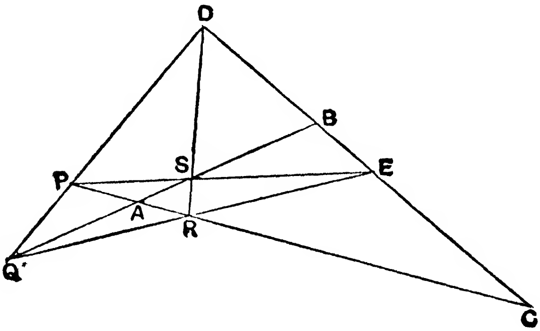
The line $(lx + my + nz) - lx = 0$ passes through the intersection of PQ and BC. Moreover it passes through A (1, 0, 0).

Therefore $my + nz = 0$ is the equation of AD.

Therefore since $A\,(DE, BC) = -1$, the equation of AE must be $my - nz = 0$.

Now the line PS passes through P the intersection of PQ and AC. Therefore its equation is of the form

$$lx + my + nz + \lambda y = 0.$$



Moreover the line PS passes through E the intersection of AE and BC. Therefore its equation is of the form

$$my - nz + \mu x = 0.$$

Whence, since these are the same line,

$$\frac{\mu}{l} = \frac{m}{m+\lambda} = \frac{-n}{n},$$

$$\therefore \mu = -l.$$

Therefore the equation of PS is $lx - my + nz = 0$.

By similar reasoning, which the student can effect for himself, it can be shewn that the equation of SR is

$$-lx + my + nz = 0,$$

and that that of QR is $lx + my - nz = 0$.

335. It is easy to see that the proposition of the preceding paragraph and that of § 331 are reciprocal to each other. For if we reciprocate the point (f, g, h) with respect to the imaginary conic $x^2 + y^2 + z^2 = 0$, we shall obtain the line $fx + gy + hz = 0$.

And the reciprocals of the points

$$(-f, g, h), \; (f, -g, h), \; (f, g, -h)$$

are $\quad -fx + gy + hz = 0, \;\; fx - gy + hz = 0, \;\; fx + gy - hz = 0.$

336. Conics touching four lines.

By a proper choice of the triangle of reference all conics touching four given lines (no three concurrent) will have their equation of the form $Ax^2 + By^2 + Cz^2 = 0$, *where the constants* A, B, C *are connected by a relation which is linear in the reciprocals of* A, B, C.

This may be proved by taking the four lines to be

$$\left.\begin{array}{r} lx + my + nz = 0 \\ -lx + my + nz = 0 \\ lx - my + nz = 0 \\ lx + my - nz = 0 \end{array}\right\} \quad \text{..................(1)},$$

and expressing the conditions that the conic

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

should touch these lines.

It will be found that these lead to the relations

$$F = G = H = 0$$

and
$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = 0.$$

But we can get the same result very easily from § 332 by reciprocation. For as the conic touches the lines (1) its reciprocal with respect to $x^2 + y^2 + z^2 = 0$ must pass through the points (l, m, n), $(-l, m, n)$, $(l, -m, n)$, $(l, m, -n)$.

But the general equation of conics through these points is (§ 332)

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 0 \quad \text{........................(2)},$$

where
$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = 0.$$

And the reciprocal of (2) is (§ 320) $Ax^2 + By^2 + Cz^2 = 0$.

Thus the general equation of conics touching the lines (1) is

$$Ax^2 + By^2 + Cz^2 = 0,$$

where the constants A, B, C, different for the different conics, are connected by the relation

$$\frac{l^2}{A} + \frac{m^2}{B} + \frac{n^2}{C} = 0.$$

Example. The locus of the centres of conics touching four given lines is a straight line.

337. Constant cross ratio property of conics.

If A, B, C, D be four fixed points on a conic, and P a variable point on it, then $P(ABCD)$ is constant.

We have seen (§ 321) that a conic can be expressed in terms of a single parameter t in the form

$$x : y : z = at^2 + bt + c : a't^2 + b't + c' : a''t^2 + b''t + c''.$$

Now let t_1, t_2, t_3, t_4 be the values of the parameter t for the four points A, B, C, D on the conic; and let τ be the value of the parameter for *any* point P on the conic.

Take any two lines through P,

$$X \equiv lx + my + nz = 0, \qquad Y \equiv l'x + m'y + n'z = 0.$$

Then any line through P will have its equation of the form

$$X - \lambda Y = 0.$$

Now let λ_1 be the special value of λ for the line AP.

Then we must have

$$l(at_1^2 + bt_1 + c) + m(a't_1^2 + b't_1 + c') + n(a''t_1^2 + b''t_1 + c'')$$
$$= \lambda_1 \{l'(at_1^2 + bt_1 + c) + m'(a't_1^2 + b't_1 + c') + n'(a''t_1^2 + b''t_1 + c'')\}$$

and

$$l(a\tau^2 + b\tau + c) + m(a'\tau^2 + b'\tau + c') + n(a''\tau^2 + b''\tau + c'')$$
$$= \lambda_1 \{l'(a\tau^2 + b\tau + c) + m'(a'\tau^2 + b'\tau + c) + n'(a''\tau^2 + b''\tau + c'')\}.$$

On subtraction, and division by $t - \tau$ we get

$$l\{a(t_1 + \tau) + b\} + m\{a'(t_1 + \tau) + b'\} + n\{a''(t_1 + \tau) + b''\}$$
$$= \lambda_1 [l'\{a(t_1 + \tau) + b\} + m'\{a'(t_1 + \tau) + b'\} + n'\{a''(t_1 + \tau) + b''\}].$$

From which we get

$$\lambda_1 = \frac{At_1 + B}{Ct_1 + D},$$

where A, B, C, D are independent of t_1.

Now $$P(ABCD) = \frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)} \text{ (§ 328)},$$

and

$$\lambda_1 - \lambda_2 = \frac{At_1 + B}{Ct_1 + D} - \frac{At_2 + B}{Ct_2 + D} = \frac{(AD - BC)(t_1 - t_2)}{(Ct_1 + D)(Ct_2 + D)}.$$

Thus we see that

$$P(ABCD) = \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_4)(t_3 - t_2)},$$

which is independent of τ.

Thus $P(ABCD)$ is constant.

338. Special case.

The following is a special case of the preceding article.

The cross ratios of the pencil formed by joining any point P on the conic $Ax^2 + By^2 + Cz^2 = 0$ to the four points $(\pm f, \pm g, \pm h)$ through which it passes are

$$-\frac{Bg^2}{Ch^2}, \quad -\frac{Ch^2}{Af^2}, \quad -\frac{Af^2}{Bg^2},$$

and their reciprocals.

Since the conic passes through the four points we have

$$Af^2 + Bg^2 + Ch^2 = 0.$$

Now the equation of the conic can be written

$$(x\sqrt{A} + y\sqrt{-B})(x\sqrt{A} - y\sqrt{-B}) = (z\sqrt{-C})^2.$$

Thus (§ 321) we may take as our parameter t either of the equal ratios

$$\frac{x\sqrt{A} + y\sqrt{-B}}{z\sqrt{-C}} \text{ and } \frac{z\sqrt{-C}}{x\sqrt{A} - y\sqrt{-B}}.$$

Whence

$$t_1 = \frac{f\sqrt{A} + g\sqrt{-B}}{h\sqrt{-C}}, \quad t_2 = \frac{-f\sqrt{A} + g\sqrt{-B}}{h\sqrt{-C}},$$

$$t_3 = \frac{f\sqrt{A} - g\sqrt{-B}}{h\sqrt{-C}}, \quad t_4 = \frac{f\sqrt{A} + g\sqrt{-B}}{-h\sqrt{-C}},$$

$$\therefore \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_4)(t_3 - t_2)} = \frac{(2f\sqrt{A})(2f\sqrt{A})}{2(f\sqrt{A} + g\sqrt{-B})\,2(f\sqrt{A} - g\sqrt{B})}.$$

$$= \frac{Af^2}{Af^2 + Bg^2} = -\frac{Af^2}{Ch^2}.$$

This then is one of the cross ratios and the others follow from symmetry.

Examples. **1.** Prove analytically that if A, B, C, D be four coplanar points of which no three are collinear, then the locus of a point P such that $P(ABCD)$ is constant is a conic through A, B, C, D.

2. Prove that the constant cross ratio of the pencil formed by joining any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the four fixed points upon it (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) is equal to

$$\frac{\left(\frac{y_1-b}{x_1}-\frac{y_2-b}{x_2}\right)\left(\frac{y_3-b}{x_3}-\frac{y_4-b}{x_4}\right)}{\left(\frac{y_1-b}{x_1}-\frac{y_4-b}{x_4}\right)\left(\frac{y_3-b}{x_3}-\frac{y_2-b}{x_2}\right)}.$$

3. Prove that one of the cross ratios of the pencil formed by joining any point on an ellipse to the extremities of the latera recta is e^2.

4. Prove that the extremities of any diameter of an ellipse are harmonically conjugate with the extremities of the diameter conjugate to it.

[It is here to be proved that if Q be any point on the ellipse and PCP', DCD' be a pair of conjugate diameters $Q(PP', DD')=-1$.]

5. Prove that the cross ratio of the pencil formed by joining any point on the parabola $y^2=4ax$ to the points $(a\mu_1^2, 2a\mu_1)$, $(a\mu_2^2, 2a\mu_2)$, $(a\mu_3^2, 2a\mu_3)$, $(a\mu_4^2, 2a\mu_4)$ is

$$\frac{(\mu_1-\mu_2)(\mu_3-\mu_4)}{(\mu_1-\mu_4)(\mu_3-\mu_2)}.$$

339. Involution.

We know that two pairs of lines through a point completely determine an involution (*Pure Geometry*, § 77). We shall now establish an analytical test that a pair of lines through a point should belong to an involution determined by two other pairs of lines through the point.

If an involution be determined by the pairs of lines through the origin, or through a vertex of the triangle of reference, viz.

$$a_1x^2+2h_1xy+b_1y^2=0 \quad\text{..................(1)},$$

$$a_2x^2+2h_2xy+b_2y^2=0 \quad\text{..................(2)},$$

the necessary and sufficient condition that the pair of lines

$$a_3x^2+2h_3xy+b_3y^2=0 \quad\text{..................(3)}$$

should belong to this involution is

$$\begin{vmatrix} a_1, & h_1, & b_1 \\ a_2, & h_2, & b_2 \\ a_3, & h_3, & b_3 \end{vmatrix}=0.$$

For let $$AX^2 + 2HXY + BY^2 = 0 \quad \text{..............}(4)$$
be the double lines of the involution.

Then (1) and (2) are both harmonically conjugate with (4) (*Pure Geometry*, § 82).

$$\therefore \quad Ba_1 + Ab_1 - 2Hh_1 = 0,$$
$$Ba_2 + Ab_2 - 2Hh_2 = 0.$$

Now if (3) belong to the involution determined by (1) and (2), then (3) must also be harmonically conjugate with (4),

$$\therefore \quad Ba_3 + Ab_3 - 2Hh_3 = 0.$$

On elimination of A, B, H we get

$$\begin{vmatrix} a_1, & h_1, & b_1 \\ a_2, & h_2, & b_2 \\ a_3, & h_3, & b_3 \end{vmatrix} = 0$$

as a necessary condition.

It is also sufficient, for supposing it to hold it will be possible to determine A, B, H to satisfy the equations

$$a_1B + b_1A - 2Hh_1 = 0,$$
$$a_2B + b_2A - 2Hh_2 = 0,$$
$$a_3B + b_3A - 2Hh_3 = 0,$$

that is to say there will be a pair of lines

$$Ax^2 + 2Hxy + By^2 = 0$$

harmonically conjugate with all three of the pairs of lines, which must therefore belong to an involution.

COR. By similar reasoning we can shew that the above is also the necessary and sufficient condition that three pairs of points on the x-axis given by

$$a_1x^2 + 2h_1x + b_1 = 0,$$
$$a_2x^2 + 2h_2x + b_2 = 0,$$
$$a_3x^2 + 2h_3x + b_3 = 0,$$

should belong to the same involution.

340. Double lines.

The equation of the double lines of the involution determined by

$$a_1x^2 + 2h_1xy + b_1y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$
$$a_2x^2 + 2h_2xy + b_2y^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

is

$$\begin{vmatrix} y^2, & -xy, & x^2 \\ a_1, & h_1, & b_1 \\ a_2, & h_2, & b_2 \end{vmatrix} = 0.$$

For let $\alpha x + \beta y = 0$ be one of the double lines. Then (1), (2) and

$$\alpha^2x^2 + 2\alpha\beta xy + \beta^2y^2 = 0$$

are three pairs of lines belonging to the same involution,

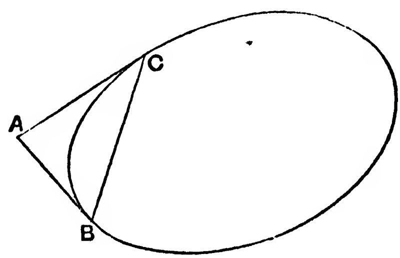
$$\therefore \begin{vmatrix} \alpha^2, & \alpha\beta, & \beta^2 \\ a_1, & h_1, & b_1 \\ a_2, & h_2, & b_2 \end{vmatrix} = 0.$$

Writing $\dfrac{\alpha}{\beta} = -\dfrac{y}{x}$ we get the equation of the double lines

$$\begin{vmatrix} y^2, & -xy, & x^2 \\ a_1, & h_1, & b_1 \\ a_2, & h_2, & b_2 \end{vmatrix} = 0.$$

341. Involution properties of conics.

Conjugate lines through a point to a conic form an involution of which the tangents from the point are the double lines.



Let A be the point, AB and AC the tangents from it. Take ABC as the triangle of reference. Thus the equation of the conic is of the form $x^2 - 2kyz = 0$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$.

Take any line through A

$$y - \lambda z = 0 \quad \text{.......................(2)}.$$

Let (x_1, y_1, z_1) be the pole of this.

Then the equation $xx_1 - k(yz_1 + y_1 z) = 0$ must be identical with (2),

$$\therefore \; x_1 = 0 \text{ and } \frac{y_1}{\lambda} = -z_1$$

Thus $y + \lambda z = 0$ is the equation of the line joining A to the pole of (2), that is $y + \lambda z = 0$ is the conjugate line to (2) through A.

And these two conjugate lines are harmonically conjugate with $y = 0$, $z = 0$, that is with AC and AB.

Therefore they belong to the involution of which AB and AC are the double lines.

COR. *Conjugate diameters of a conic form an involution system, the asymptotes being the double lines.*

342. Test for conjugate diameters.

By means of the proposition just established we can prove that *the condition that the pair of lines* (*in Cartesian coordinates*)

$$ax^2 + 2hxy + by^2 = 0 \quad \text{..................(1)}$$

should be conjugate diameters of the conic

$$Ax^2 + 2Hxy + By^2 = 1 \quad \text{..............(2)}$$

is

$$Ab + Ba - 2Hh = 0.$$

For if (1) be conjugate diameters of (2) they must belong to the involution of which the asymptotes (*i.e.* the tangents from the centre) are the double lines.

But the equation of the asymptotes is

$$Ax^2 + 2Hxy + By^2 = 0 \text{..................(3)}.$$

Thus (1) and (3) must be harmonically conjugate,

$$\therefore \; Ab + Ba - 2Hh = 0.$$

This condition then is necessary, and it is easily seen to be sufficient.

343. If we reciprocate the theorem of § 341 we get:

Conjugate points on a line with respect to a conic form an involution range the double points of which are the points in which the line cuts the conic.

This is proved in *Pure Geometry*, § 92. We will give an independent analytical proof of it as follows.

Let the line on which the points lie cut the conic in A and B (these points may be imaginary). Take the middle point O of AB for origin, OA for the axis of x and a line perpendicular to it for the y-axis.

The equation of the conic will therefore be of the form

$$ax^2 + 2hxy + by^2 + 2fy + c = 0,$$

there being no term of the first order in x since the values of x when $y = 0$ have to be equal in magnitude and opposite in sign.

Let P and P' be a pair of conjugate points situated on the given line, $(x_1, 0)$ $(x_1', 0)$ their coordinates. Then the polar of each goes through the other. Now the polar of $(x_1, 0)$ is

$$axx_1 + hx_1y + fy + c = 0,$$

$$\therefore\ ax_1'x_1 + c = 0,$$

$$\therefore\ x_1x_1' = -\frac{c}{a},$$

which is constant and $= OA^2 = OB^2$.

Therefore P and P' belong to an involution of which O is the centre and A, B are the double points.

344. Proposition. *A system of conics through four fixed points is cut by any transversal in pairs of points which form an involution.*

Take the line of the transversal for the axis of x.

Let $\qquad S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ......(1),

$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0 \text{ ......(2)}$$

be two conics through the four points, then any other conic through these points will be of the form $S + \lambda S' = 0$ ......(3).

Now the pair of points in which the transversal (which we have taken for the x-axis) cuts (1) is given by

$$ax^2 + 2gx + c = 0.$$

So the pairs of points in which the transversal cuts (2) and (3) are given by

$$a'x^2 + 2g'x + c' = 0$$

and $$(a + \lambda a')\,x^2 + 2\,(g + \lambda g')\,x + (c + \lambda c') = 0,$$

and all these belong to the same involution since

$$\begin{vmatrix} a, & g, & c \\ a', & g', & c' \\ a + \lambda a', & g + \lambda g', & c + \lambda c' \end{vmatrix} = 0.$$

Thus the proposition is proved.

345. Homographic ranges.

If x and x' be the x-coordinates of two points P and P' on the x-axis, and x, x' be connected by the relation

$$Axx' + Hx + H'x' + B = 0,$$

where A, B, H, H' are any constants, then a range of points typified by P will be homographic with a range typified by P'.

For let P_1, P_2, P_3, P_4 be any four points of the P series of points, and let x_1, x_2, x_3, x_4 be their x-coordinates.

Let P_1', P_2', P_3', P_4' be the corresponding points in the P' series, and let x_1', etc. be their x-coordinates.

Then $$(P_1'P_2'P_3'P_4') = \frac{(x_1' - x_2')(x_3' - x_4')}{(x_1' - x_4')(x_3' - x_2')}$$

$$= \frac{\left(-\dfrac{Hx_1 + B}{Ax_1 + H'} + \dfrac{Hx_2 + B}{Ax_2 + H'}\right)\left(-\dfrac{Hx_3 + B}{Ax_3 + H'} + \dfrac{Hx_4 + B}{Ax_4 + H'}\right)}{\left(-\dfrac{Hx_1 + B}{Ax_1 + H'} + \dfrac{Hx_4 + B}{Ax_4 + H'}\right)\left(-\dfrac{Hx_3 + B}{Ax_3 + H'} + \dfrac{Hx_2 + B}{Ax_2 + H'}\right)}$$

$$= \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)} = (P_1P_2P_3P_4).$$

Thus the proposition is proved.

346. The above relation gives

$$x' = -\frac{Hx + B}{Ax + H'} \quad\text{and}\quad x = -\frac{H'x' + B}{Ax' + H}.$$

Now suppose that $P(x)$ and $Q(x')$ are any two points on the line connected by this relation.

Then to P, regarded as belonging to the first series of points, corresponds Q in the second series.

But the point Q regarded as belonging to the first series will correspond with some other point in the other series.

But in the special case where $H = H'$ we have

$$x' = -\frac{Hx + B}{Ax + H}$$

and

$$x = -\frac{Hx' + B}{Ax' + H},$$

so that P considered as belonging to the first series will correspond with Q in the second series, and at the same time Q considered as belonging to the first series will correspond with P in the second series.

Thus if P, P'; Q, Q'; R, R', be three pairs of corresponding points in the two series,

$$(PQRP') = (P'Q'R'P).$$

That is these three pairs of points belong to an involution (*Pure Geometry*, § 80).

This we can see analytically thus:

We have $\quad Axx' + H(x + x') + B = 0.$

If then the pair of points $P(x)$, $P'(x')$ be given by

$$aX^2 + 2hX + b = 0,$$

we have

$$x + x' = -\frac{2h}{a}$$

and

$$xx' = \frac{b}{a},$$

$$\therefore\ Ab + Ba = 2Hh,$$

that is to say the pair of points P, P' is harmonically conjugate with the pair given by $AX^2 + 2HX + B = 0$.

Therefore all the pairs P, P'; Q, Q'; etc. being harmonically conjugate with the same two points in the line, form an involution with these two points as the double points.

347. We can also see that pairs of points connected by the relation

$$Axx' + H(x + x') + B = 0$$

form an involution in the following way:

Transform the origin to the point $(k, 0)$. Let the new x-coordinates of the two points be X, X' so that

$$x = X + k, \quad x' = X' + k,$$

$$\therefore\ A(X + k)(X' + k) + H(X + X' + 2k) + B = 0,$$

$$\therefore\ AXX' + (Ak + H)(X + X') + Ak^2 + 2Hk + B = 0.$$

Now choose k so that $Ak + H = 0$.

Then we have

$$AXX' = \frac{H^2}{A} - B,$$

i.e.

$$XX' = \frac{H^2 - AB}{A^2}.$$

Thus the pairs of points belong to an involution whose centre is the new origin and whose radius is $\sqrt{H^2 - AB}/A$. Hence the double points of the involution will be real or imaginary according as H^2 is $>$ or $<$ AB.

348. Double points.

Now let us return to the more general relation

$$Axx' + Hx + H'x' + B = 0,$$

defining two homographic ranges which do not make an involution.

It is clear that there will be two points which will correspond to themselves in the two series. For putting $x' = x$ we have

$$Ax^2 + (H + H')x + B = 0.$$

The points so given are called the *double points* of the homographic ranges.

If we denote the double points by Ω and Ω', and P, P' be any pair of corresponding points $(\Omega\Omega' PP')$ is constant.

For if Q and Q' be another pair of corresponding points,

$$(\Omega\Omega' PQ) = (\Omega\Omega' P'Q'),$$

$$\therefore\ (\Omega\Omega' PP') = (\Omega\Omega' QQ'),$$

that is $(\Omega\Omega' PP')$ is constant.

In proving this we have made use of the theorem that if

$$(ABCD) = (ABC'D'), \quad \text{then } (ABCC') = (ABDD').$$

This is easily seen to be the case; for if

$$(ABCD) = (ABC'D'),$$

then
$$(ACBD) = (AC'BD'),$$

by interchanging the second and third letters in both.

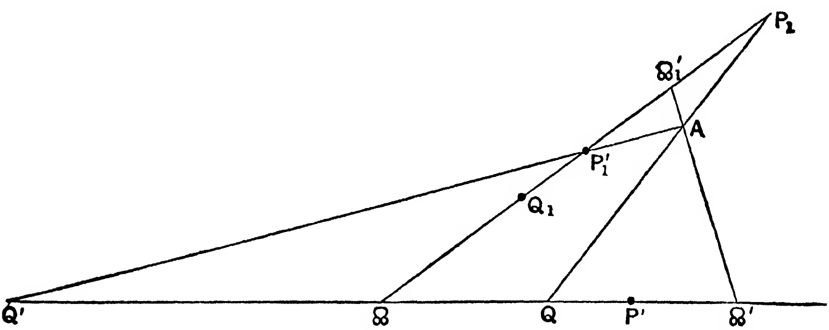
$$\therefore\ \frac{AC.BD}{AD.BC} = \frac{AC'.BD'}{AD'.BC'},$$

$$\therefore\ \frac{AC.BC'}{AC'.BC} = \frac{AD.BD'}{AD'.BD},$$

$$\therefore\ (ACBC') = (ADBD'),$$

$$\therefore\ (ABCC') = (ABDD').$$

349. The constancy of $(\Omega\Omega' PP')$ may be made use of to find the point corresponding to a point Q, when the double points Ω, Ω' and a pair of points P and P' are known.



For let the line be turned round Ω through any angle and let P_1, Q_1, P_1', Q_1', be the new positions of P_1, Q_1, etc.

Now $(\Omega\Omega'QQ') = (\Omega\Omega'PP') = (\Omega\Omega_1'P_1P_1')$.

Therefore since $\Omega\Omega'QQ'$ and $\Omega\Omega_1'P_1P_1'$ are homographic and have a common point Ω, $\Omega'\Omega_1'$, P_1Q, $P_1'Q'$ are concurrent (*Pure Geometry*, § 60).

Hence to find Q' corresponding to Q, we let $\Omega'\Omega_1'$, and P_1Q meet in A and then join $P_1'A$ to meet the given line in Q'.

350. We have seen that points in a line connected by the relation

$$Axx' + Hx + H'x' + B = 0$$

give two homographic ranges. We shall now shew that if there be two homographic ranges in a line corresponding points are connected by a relation of the above form.

For let a, b, c be the distances from the origin of three points A, B, C in the line and let a', b', c' be the distances of the corresponding points A', B', C'.

Let x and x' be the distances of any other pair of corresponding points.

Then since $(PABC) = (P'A'B'C')$,

$$\therefore \frac{(x-a)(b-c)}{(x-c)(b-a)} = \frac{(x'-a')(b'-c')}{(x'-c')(b'-a')},$$

and this reduces to the form

$$Axx' + Hx + H'x' + B = 0.$$

351. Next let it be observed that it is not necessary that the points P, Q, etc. of the one series be measured from the same point O as that from which the corresponding points P', Q', etc. are measured.

Say that the P series is measured from O, and the P' series from O', and $OP = x$, $O'P' = x'$, then x and x' are still connected by a similar relation.

For let $OO' = h$,

$$\therefore OP' = OO' + O'P' = h + x'.$$

Now $Ax \, . \, OP' + Hx + H' \, . \, OP' + B = 0$,

$$\therefore Ax(h + x') + Hx + H'(h + x') + B = 0,$$

and this is still of the form

$$Axx' + Hx + H'x' + B = 0$$

with changed constants.

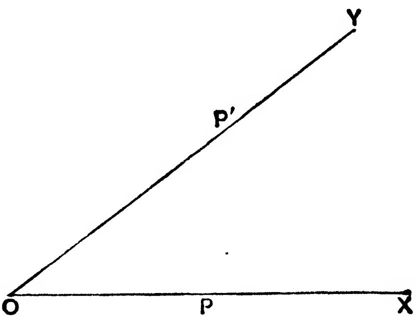
352. Again, we can see that it is not necessary that the system of points P', etc. should be collinear with the P system.

If x and x' be the distances of P and P' measured from points O and O' in the respective lines of the ranges, the system P will be homographic with the system P' provided a relation of the form

$$Axx' + Hx + H'x' + B = 0$$

obtains, and conversely.

We can derive at once the property referred to in § 349 that *if two homographic ranges lie on lines which meet, and the point of intersection of the lines correspond to itself in the two ranges, the lines joining corresponding points are all concurrent.*



For take the two lines OX, OY on which the ranges lie as axes of coordinates.

Let x and y be the distances of corresponding points from O,

$$\therefore\ Axy + Hx + H'y + B = 0,$$

and as O is a corresponding point in the two by hypothesis, this relation must be satisfied by $x = 0$, $y = 0$, $\therefore B = 0$, and the relation is

$$Axy + Hx + H'y = 0.$$

Now let p and p' be the distances from O of two corresponding points P and P',

$$\therefore\ App' + Hp + H'p' = 0,$$

$$\therefore\ \frac{H}{p'} + \frac{H'}{p} = -A.$$

Thus the line PP' whose equation is $\dfrac{x}{p} + \dfrac{y}{p'} = 1$ passes through the fixed point $(-H'/A,\ -H/A)$.

We shall prove in a later chapter that the lines joining corresponding points of two homographic ranges on intersecting lines, the point of intersection of which does *not* correspond to itself in the two ranges, all touch a conic.

EXAMPLES.

1. Prove that the middle points of the three diagonals of the quadrilateral formed by the lines

$$\pm lx \pm my \pm nz = 0$$

lie on the line

$$l^2x + m^2y + n^2z = 0.$$

2. Prove analytically that the locus of the poles of a given line with respect to conics passing through four fixed points is a conic which passes through the diagonal points of the quadrangle formed by the given points.

3. State and give an analytical proof of the reciprocal of the theorem of Ex. 2.

4. The director circles of all conics touching four given straight lines form a coaxial system.

5. If a system of conics have a common self-conjugate triangle, any line through one of the vertices is cut by the system in pairs of points which form an involution.

6. Shew that two of the family of conics drawn through four fixed points, no three of which are collinear, are parabolas; and that the necessary and sufficient condition that these parabolas should be real is that the triangle formed by joining *any* three of the points should not enclose the fourth.

7. A system of four point conics can be reciprocated into concentric conics.

8. Prove that the four common tangents to the conics

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=0$$

and
$$\frac{x^2}{A'}+\frac{y^2}{B'}+\frac{z^2}{C'}=0$$

are
$$x\sqrt{BC'-B'C}\pm y\sqrt{CA'-C'A}\pm z\sqrt{AB'-A'B}=0,$$

and that all conics which touch these four lines are included in

$$\frac{x^2}{A+kA'}+\frac{y^2}{B+kB'}+\frac{z^2}{C+kC'}=0.$$

9. Prove that the locus of points from which the pair of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are harmonic conjugates to the pair of parallels to the asymptotes is the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2.$$

10. The tangents from a point P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ are harmonic conjugates with respect to the tangents from P to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{c^2}+1=0$; shew that the locus of P consists of a pair of parallel lines.

11. From a point P on a parabola two normals are drawn to the curve. Prove that the bisectors of the angles between these, with the diameter through P and the normal at P form a harmonic pencil.

12. Prove that the four points on an ellipse whose eccentric angles are α, β, γ, δ subtend at any point on the ellipse a pencil whose cross ratio is

$$\frac{\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\delta}{2}}{\sin\frac{\alpha-\delta}{2}\sin\frac{\gamma-\beta}{2}}.$$

13. The double points of the involution determined by the two pairs of points given by the quadratics

$$ax^2 + 2bx + c = 0, \quad Ax^2 + 2Bx + C = 0,$$

are given by

$$(aB - bA)x^2 - (cA - aC)x + (bC - cB) = 0.$$

PP', QQ', RR' are six points on a straight line. Prove that the six double points of the three involutions determined by QQ', RR'; RR', PP'; and PP', QQ' cannot be in involution unless PP', QQ', RR' are in involution, when the double points coincide in three.

14. Shew that the complete condition that the pairs of lines

$$ax^2 + 2bxy + cy^2 = 0 \text{ and } a'x^2 + 2b'xy + c'y^2 = 0$$

should form a harmonic pencil is

$$(a'c + ac' - 2bb')\{(a'c + ac' - 2bb')^2 - 36(ac - b^2)(a'c' - b'^2)\} = 0.$$

15. Through the angular point A of the triangle of reference a straight line AD is drawn, cutting the conic

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

in the points P and P', and also cutting the line

$$L \equiv lx + my + nz = 0$$

in the point Q. If a point Q' be taken on AD such that the range (PP', QQ') is harmonic, shew that as the line AD moves, the locus of Q' is the conic

$$lS = L(ax + hy + gz).$$

16. Four points A, B, C, D are taken on $\dfrac{l}{r} = 1 + e\cos\theta$, and the corresponding values of θ are α, β, γ, δ. Shew that the cross ratio of the pencil subtended by A, B, C, D at any point of the conic is the ratio with the sign changed of some two of the three quantities

$$\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\delta}{2}, \quad \sin\frac{\gamma-\alpha}{2}\sin\frac{\beta-\delta}{2}, \quad \sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\delta}{2}.$$

17. A square is inscribed in an ellipse whose semi-axes are a and b, and any point on the ellipse is joined to the corners of the square. Prove that one of the cross ratios of the pencil so formed is $-\dfrac{b^2}{a^2}$.

18. The locus of points from which the two pairs of tangents to the circles

$$(x-a)^2+y^2=c^2, \quad (x+a)^2+y^2=c^2$$

form conjugate pairs of a harmonic pencil is

$$(2a^2-c^2)\,y^2-c^2x^2=c^2\,(a^2-c^2).$$

19. If two complete quadrilaterals have the same lines for diagonals their eight sides touch a conic.

20. Given four points on a conic section, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point.

21. If l_1, l_2 be the lines $ax^2+2hxy+by^2=0$, and l_3, l_4 the lines $a'x^2+2h'xy+b'y^2=0$, and λ be the cross ratio of the pencil l_1, l_3, l_2, l_4, then

$$\left(\frac{\lambda+1}{\lambda-1}\right)^2=\frac{(ab'+a'b-2hh')^2}{4\,(ab-h^2)\,(a'b'-h'^2)}.$$

22. Shew that the three lines each of which forms a harmonic pencil with the lines

$$y=0, \quad ax^2+2hxy+by^2=0$$

are

$$ax+hy=0,$$

$$ax^2+2hxy+\left(9b-\frac{8h^2}{a}\right)y^2=0.$$

23. If $(a, b, c, f, g, h)(x, y, 1)^2=1$ be the Cartesian equation of a conic and if $X\equiv ax+hy+g$, $Y=hx+by+f$, prove that the equation of the asymptotes is

$$bX^2-2hXY+aY^2=0.$$

Hence shew that the lines $b'X^2-2h'XY+a'Y^2=0$ will be a pair of conjugate diameters if $ab'+a'b-2hh'=0$.

CHAPTER XVII.

INVARIANTS.

353. It follows from what has been proved in § 106 that if (x, y) be the Cartesian coordinates of a point referred to axes including an angle ω and if

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

transforms into

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x + 2f'y + c',$$

where (x', y') are the coordinates of the same point referred to any other axes in the plane,

$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} \text{ and } \frac{ab - h^2}{\sin^2\omega}$$

are invariant, for by a change of *origin* the coefficients of the terms $ax^2 + 2hxy + by^2$ are unchanged, and by a change in the *direction of the axes* the invariant relation has been shewn to be true. There is another invariant relation between the coefficients a, b, c, f, g, h which we have not yet given. We shall before proving it establish the following proposition.

Proposition. *If the homogeneous coordinates x, y, z be transformed to new coordinates x', y', z' by the substitutions*

$$\left.\begin{aligned} x &= l_1x' + m_1y' + n_1z' \\ y &= l_2x' + m_2y' + n_2z' \\ z &= l_3x' + m_3y' + n_3z' \end{aligned}\right\} \text{..................(1)},$$

and $$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \text{ ....(2)}$$

be transformed to

$$a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y' \text{......(3)},$$

then
$$\begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix} = \epsilon^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

where
$$\epsilon \equiv \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}.$$

On making the substitutions* (1) we see that (2) becomes

$$a(l_1x' + m_1y' + n_1z')^2 + b(l_2x' + m_2y' + n_2z')^2 + c(l_3x' + m_3y' + n_3z')^2$$
$$+ 2f(l_2x' + m_2y' + n_2z')(l_3x' + m_3y' + n_3z') + 2g(l_3x' + m_3y' + n_3z')$$
$$\times (l_1x' + m_1y' + n_1z') + 2h(l_1x' + m_1y' + n_1z')(l_2x' + m_2y' + n_2z').$$

Comparing this with (3) we have

$$a' = al_1^2 + bl_2^2 + cl_3^2 + 2fl_2l_3 + 2gl_3l_1 + 2hl_1l_2,$$
$$b' = am_1^2 + bm_2^2 + \text{etc.},$$
$$c' = an_1^2 + bn_2^2 + \text{etc.},$$
$$f' = am_1n_1 + bm_2n_2 + cm_3n_3 + f(m_2n_3 + m_3n_2) + g(m_3n_1 + m_1n_3) + h(m_1n_2 + m_2n_1),$$
$$g' = an_1l_1 + bn_2l_2 + cn_3l_3 + f(n_2l_3 + n_3l_2) + g(n_3l_1 + n_1l_3) + h(n_1l_2 + n_2l_1),$$
$$h' = al_1m_1 + bl_2m_2 + cl_3m_3 + f(l_2m_3 + l_3m_2) + g(l_3m_1 + l_1m_3) + h(l_1m_2 + l_2m_1).$$

If now we write

$$\left.\begin{aligned} L_1 &= al_1 + hl_2 + gl_3 \\ L_2 &= hl_1 + bl_2 + fl_3 \\ L_3 &= gl_1 + fl_2 + cl_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} M_1 &= am_1 + hm_2 + gm_3 \\ M_2 &= hm_1 + bm_2 + fm_3 \\ M_3 &= gm_1 + fm_2 + cm_3 \end{aligned}\right\},$$

$$\left.\begin{aligned} N_1 &= an_1 + hn_2 + gn_3 \\ N_2 &= hn_1 + bn_2 + fn_3 \\ N_3 &= gn_1 + fn_2 + cn_3 \end{aligned}\right\},$$

we have

$$a' = l_1L_1 + l_2L_2 + l_3L_3,$$
$$b' = m_1M_1 + m_2M_2 + m_3M_3,$$
$$c' = n_1N_1 + n_2N_2 + n_3N_3,$$

* See the Cambridge Tract on *Quadratic Forms* by T. J. I'A. Bromwich, Sc.D. (*Camb. University Press*).

$$f' = m_1N_1 + m_2N_2 + m_3N_3 = n_1M_1 + n_2M_2 + n_3M_3,$$
$$g' = n_1L_1 + n_2L_2 + n_3L_3 = l_1N_1 + l_2N_2 + l_3N_3,$$
$$h' = l_1M_1 + l_2M_2 + l_3M_3 = m_1L_1 + m_2L_2 + m_3L_3.$$

$$\therefore \begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix} = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \times \begin{vmatrix} L_1, & M_1, & N_1 \\ L_2, & M_2, & N_2 \\ L_3, & M_3, & N_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \times \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \times \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

$$= \epsilon^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

'354. We can use the proposition of the preceding paragraph to prove that if

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$

where x, y refer to some Cartesian axes inclined at an angle ω, be transformed to

$$a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x + 2f'y' + c',$$

where x', y' refer to Cartesian axes inclined at an angle ω' with the same or a different origin, then

$$\frac{\begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix}}{\sin^2\omega'} = \frac{\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}}{\sin^2\omega}.$$

For we make our expressions homogeneous by the insertion of z and z' which are unity, then

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

transforms into $a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y'$ by substitutions of the form

$$\begin{aligned} x &= l_1x' + m_1y' + n_1z', \\ y &= l_2x' + m_2y' + n_2z', \\ z &= z', \end{aligned}$$

so in this case

$$\epsilon = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ 0, & 0, & 1 \end{vmatrix} = (l_1 m_2 - l_2 m_1).$$

Now we know that the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\tfrac{1}{2} \sin \omega \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1 & z_2, & z_3 \end{vmatrix}$$

$$= \tfrac{1}{2} \sin \omega \begin{vmatrix} l_1 x_1' + m_1 y_1' + n_1 z_1', & l_1 x_2' + m_2 y_2' + n_2 z_2', & l_1 x_3' + m_1 y_3' + n_1 z_3' \\ l_2 x_1' + m_2 y_1' + n_2 z_1', & l_2 x_2' + m_2 y_2' + n_2 z_2', & l_2 x_3' + m_2 y_3' + n_3 z_3' \\ z_1', & z_2', & z_3' \end{vmatrix}$$

$$= \tfrac{1}{2} \sin \omega \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_3', & y_3', & z_3' \end{vmatrix}.$$

But the area of the triangle is

$$\tfrac{1}{2} \sin \omega' \begin{vmatrix} x_1', & y_1', & z_1' \\ x_2', & y_2', & z_2' \\ x_3', & y_3', & z_3' \end{vmatrix},$$

$$\therefore \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ 0, & 0, & 1 \end{vmatrix} = \frac{\sin \omega'}{\sin \omega},$$

$$\therefore \frac{\Delta'}{\sin^2 \omega'} = \frac{\Delta}{\sin^2 \omega} \text{ where}$$

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

We thus see that $\dfrac{\Delta}{\sin^2 \omega}$ is an invariant for the general equation of the second degree in Cartesian axes for any changes of origin and axes.

Examples. 1. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ be a parabola in oblique Cartesian coordinates, prove that the semi-latus rectum is

$$\sin^2\omega\,(-\Delta)^{\frac{1}{2}} \div (a+b-2h\cos\omega)^{\frac{3}{2}}$$

where Δ is the discriminant $abc+2fgh-af^2-bg^2-ch^2$.

[We know that when we transform to the axis and tangent at the vertex as axes of coordinates the equation of the parabola becomes

$$y'^2-2lx'=0,$$

where l is the semi-latus rectum.

Thus we must have

$$ax^2+2hxy+by^2+2gx+2fy+c \equiv b'y'^2+2g'x'$$

where $l=-\dfrac{g'}{b'}$, that is $l^2=\dfrac{g'^2}{b'^2}$.

Now $\Delta'=-b'g'^2$

and $$\frac{\Delta'}{\Delta}=\frac{\sin^2\omega'}{\sin^2\omega}=\frac{1}{\sin^2\omega}\quad(\S\ 354),$$

$$\therefore\ \Delta=-b'g'^2\sin^2\omega=-l^2b'^3\sin^2\omega.$$

But $$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{b'}{\sin^2\frac{1}{2}\pi}=b',$$

$$\therefore\ l^2=\frac{-\Delta}{b'^3\sin^2\omega}=\frac{-\Delta\sin^4\omega}{(a+b-2h\cos\omega)^3},$$

$$\therefore\ l=\frac{(-\Delta)^{\frac{1}{2}}\sin^2\omega}{(a+b-2h\cos\omega)^{\frac{3}{2}}}.\ ]$$

2. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ be a parabola the length of the focal chord which makes an angle θ with the axis of the parabola is

$$2\sin^2\omega\,(-\Delta)^{\frac{1}{2}}\div\sin^2\theta\,(a+b-2h\cos\omega)^{\frac{3}{2}}.$$

3. If $S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represent an ellipse, prove that the difference of the eccentric angles of the points of contact of tangents from (x, y) to the curve is

$$2\tan^{-1}\left\{\frac{(h^2-ab)\,S}{\Delta}\right\}^{\frac{1}{2}}.$$

4. If the general equation in rectangular axes represent a rectangular hyperbola its equation referred to its asymptotes is

$$2\,(h^2-ab)^{\frac{3}{2}}\,xy=\Delta.$$

Investigate also the corresponding equation when the original axes of coordinates are oblique.

355. Invariants of two conics.

Let $$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots(1),$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0 \ldots(2),$$
be the equations of two conics in any system of homogeneous coordinates. Then

$$S + kS' \equiv (a + ka')\,x^2 + (b + kb')\,y^2 + (c + kc')\,z^2$$
$$+ 2\,(f + kf')\,yz + 2\,(g + kg')\,zx + 2\,(h + kh')\,zx = 0 \quad \ldots(3)$$
represents for different constant values of k a conic passing through the four points of intersection of (1) and (2).

Now let k be so chosen that (3) is a pair of straight lines, that is k satisfies

$$\begin{vmatrix} a + ka', & h + kh', & g + kg' \\ h + kh', & b + kb', & f + kf' \\ g + kg', & f + kf', & c + kc' \end{vmatrix} = 0,$$

that is

$$(a + ka')(b + kb')(c + kc') + 2\,(f + kf')(g + kg')(h + kh')$$
$$- (a + ka')(f + kf')^2 - (b + kb')(g + kg')^2$$
$$- (c + kc')(h + kh')^2 = 0,$$

that is $$\Delta' k^3 + \Theta' k^2 + \Theta k + \Delta = 0,$$
where

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

$$\Delta' \equiv a'b'c' + 2f'g'h' - af'^2 - bg'^2 - ch'^2 \equiv \begin{vmatrix} a', & h', & g' \\ h', & b', & f' \\ g', & f', & c' \end{vmatrix},$$

$$\Theta \equiv (bc - f^2)\,a' + (ca - g^2)\,b' + (ab - h^2)\,c'$$
$$+ 2\,(gh - af)\,f' + 2\,(hf - bg)\,g' + 2\,(fg - ch)\,h'$$
$$\equiv Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$

$$\Theta' \equiv (b'c' - f'^2)\,a + (c'a' - g'^2)\,b + (a'b' - h'^2)\,c$$
$$+ 2\,(g'h' - a'f')\,f + 2\,(h'f' - b'g')\,g + 2\,(f'g' - c'h')\,h$$
$$\equiv A'a + B'b + C'c + 2F'f + 2G'g + 2H'h,$$

where A, B, C, F, G, H are the minors taken with their proper sign of a, b, c, f, g, h in the determinant Δ, and A', B', etc. are the corresponding minors in Δ'.

There will thus be three values of k for which the conic (3) will represent a pair of straight lines. We can see that this must be so, for if the conics (1) and (2) intersect in P, Q, R, S, we can have three pairs of lines through these points, viz. PQ, RS; PR, QS; PS, QR; and each of these pairs of lines will be a conic through the four points.

Now suppose we transform to any other homogeneous system; let S become $\bar{S}$, and S' become $\bar{S}'$, then $S+kS'$ will become $\bar{S}+k\bar{S}'$, and exactly the same values of k will make $\bar{S}+k\bar{S}'=0$ a pair of straight lines as will make $S+kS'=0$ a pair of straight lines.

It follows then that

$$\frac{\Theta'}{\Delta'}=\frac{\bar{\Theta}'}{\bar{\Delta}'};\quad \frac{\Theta}{\Delta'}=\frac{\bar{\Theta}}{\bar{\Delta}'};\quad \frac{\Delta}{\Delta'}=\frac{\bar{\Delta}}{\bar{\Delta}'}.$$

In other words the ratio of each of the quantities Δ, Δ', Θ, Θ' to any one of the four is invariant.

These four quantities are called invariants of the two conics. But it must be clearly understood that they are not invariant individually, but only in ratios.

We have already seen (§ 353) that

$$\bar{\Delta}=\epsilon^2\Delta, \qquad \bar{\Delta}'=\epsilon^2\Delta'.$$

As then $$\frac{\bar{\Theta}}{\bar{\Delta}}=\frac{\Theta}{\Delta},$$

we must have $$\bar{\Theta}=\epsilon^2\Theta,$$

and similarly also $$\bar{\Theta}'=\epsilon^2\Theta'.$$

356. Invariants and projection.

Suppose now that we transform the equations of two conics

$$S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$
$$S'\equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0,$$

by the substitutions

$$x = \frac{\alpha X + \beta Y + \gamma Z}{\lambda X + \mu Y + \nu Z},$$

$$y = \frac{\alpha' X + \beta' Y + \gamma' Z}{\lambda X + \mu Y + \nu Z},$$

$$z = \frac{\alpha'' X + \beta'' Y + \gamma'' Z}{\lambda X + \mu Y + \nu Z},$$

the denominator being the same in all of these.

On substitution we find

$$\begin{aligned} &S(\lambda X + \mu Y + \nu Z)^2 \\ &= a(\alpha X + \beta Y + \gamma Z)^2 + b(\alpha' X + \beta' Y + \gamma' Z)^2 + c(\alpha'' X + \beta'' Y + \gamma'' Z)^2 \\ &\quad + 2f(\alpha' X + \beta' Y + \gamma' Z)(\alpha'' X + \beta'' Y + \gamma'' Z) \\ &\quad + 2g(\alpha'' X + \beta'' Y + \gamma'' Z)(\alpha X + \beta Y + \gamma Z) \\ &\quad + 2h(\alpha X + \beta Y + \gamma Z)(\alpha' X + \beta' Y + \gamma' Z) \\ &= (\text{say})\ AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY. \end{aligned}$$

And $S'(\lambda X + \mu Y + \nu Z)^2$ will be of a similar form

$$A'X^2 + B'Y^2 + C'Z^2 + 2F'YZ + 2G'ZX + 2H'XY.$$

Now if P be the point (x, y, z), and P' the point whose coordinates referred to some triangle or other be (X, Y, Z), then as P describes the locus

$$S + kS' = 0 \qquad \text{.......................(1)},$$

P' will describe the locus

$$\begin{aligned} &AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY \\ &+ k(A'X^2 + B'Y^2 + C'Z^2 + 2F'YZ + 2G'ZX + 2H'XY) = 0 \quad (2). \end{aligned}$$

And if (1) be two lines, viz.

$$(lx + my + nz)(l'x + m'y + n'z) = 0,$$

(2) will be

$$\begin{aligned} &\{l(\alpha X + \beta Y + \gamma Z) + m(\alpha' X + \beta' Y + \gamma' Z) + n(\alpha'' X + \beta'' Y + \gamma'' Z)\} \\ &\quad \times \{l'(\alpha X + \beta Y + \gamma Z) + m'(\alpha' X + \beta' Y + \gamma' Z) \\ &\qquad\qquad + n'(\alpha'' X + \beta'' Y + \gamma'' Z)\} = 0, \end{aligned}$$

which is a pair of straight lines too.

That is to say, the same values of k will make both (1) and (2) represent a pair of straight lines. Hence the invariant character of the ratios of Δ, Δ', Θ, Θ' to each other is still preserved for transformations of the form we are considering.

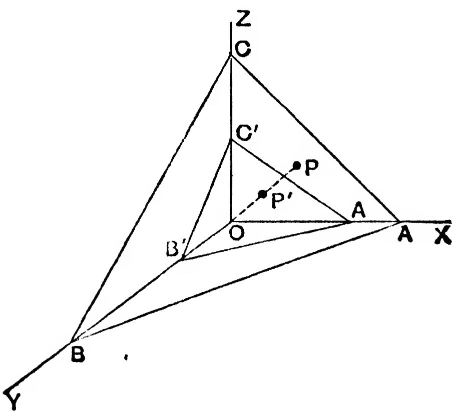
357. Now the transformation we have made in the preceding article is just that which occurs in the case where we project conically from one plane p on to another π (see *Pure Geometry*, Chap. IV.).

This we shall now proceed to shew, making use of the elements of analytical Solid Geometry.

Let O be the vertex of projection. Take three mutually perpendicular lines OX, OY, OZ cutting the p plane in A, B, C and cutting the π plane in A', B', C'. Let P be any point in the p plane, and let OP cut the π plane in P', so that P' is the projection of P.

Let (x, y, z) be the areal coordinates of P referred to the triangle ABC, and let (X, Y, Z) be those of P' referred to $A'B'C'$.

Let $OA = a$, $OB = b$, $OC = c$; and let $OA' = a'$, $OB' = b'$, $OC' = c'$.



Let (ξ, η, ζ) and (ξ', η', ζ') be the Cartesian coordinates of P and P' respectively referred to OX, OY, OZ.

Now $x = \dfrac{\Delta PBC}{\Delta ABC} = \dfrac{\text{projection of } \Delta PBC \text{ on plane } YOZ}{\text{projection of } \Delta ABC \text{ on plane } YOZ}$

$$= \begin{vmatrix} \eta, & b, & 0 \\ \zeta, & 0, & c \\ 1, & 1, & 1 \end{vmatrix} \div \begin{vmatrix} 0, & b, & 0 \\ 0, & 0, & c \\ 1, & 1, & 1 \end{vmatrix} = 1 - \frac{\eta}{b} - \frac{\zeta}{c} = \frac{\xi}{a}.$$

Thus
$$\xi = ax, \quad \eta = by, \quad \zeta = cz,$$
$$\xi' = a'X, \quad \eta' = b'Y, \quad \zeta' = c'Z.$$

But $\dfrac{\xi}{\xi'} = \dfrac{\eta}{\eta'} = \dfrac{\zeta}{\zeta'}$ for each of these ratios $= \dfrac{OP}{OP'}$,

$$\therefore \frac{ax}{a'X} = \frac{by}{b'Y} = \frac{cz}{c'Z},$$

$$\therefore \frac{x}{\frac{a'}{a}X} = \frac{y}{\frac{b'}{b}Y} = \frac{z}{\frac{c'}{c}Z} = \frac{1}{\frac{a'}{a}X + \frac{b'}{b}Y + \frac{c'}{c}Z}.$$

358. It follows from the two preceding articles that *if $S=0$, $S'=0$ be two conics in a plane whose equations are expressed in terms of any homogeneous coordinates referred to a triangle in the plane, and if these conics be projected on to another plane and the equations of the new conics be $\bar{S}=0$, $\bar{S}'=0$ referred to any triangle in the new plane, the ratios of the quantities Δ, Δ', Θ, Θ' are unaltered.*

For we can pass from the homogeneous coordinates in terms of which S and S' are expressed to areal coordinates referred to the triangle ABC of § 357, still preserving the invariant character of our ratios. We now project and get two new conics expressed in areal coordinates referred to the triangle $A'B'C'$; and by the two preceding articles the invariants still hold good, nor are they disturbed when we pass to any other homogeneous coordinates referred to any triangle in the π plane.

359. It seems desirable to give a word of caution at this point. We know that two conics can be simultaneously projected into circles by projecting two of their points of intersection into

the circular points at infinity (*Pure Geometry*, § 246). It might then seem that we could take

$$\bar{S} \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1$$

and $$\bar{S}' \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2.$$

But this would be incorrect and would lead to serious error. What we may do is to write

$$\bar{S} \equiv a_1(x^2 + y^2) + 2g_1x + 2f_1y + c_1,$$

$$\bar{S}' \equiv a_2(x^2 + y^2) + 2g_2x + 2f_2y + c_2.$$

We may then write $x\sqrt{a_1} = X$, $y\sqrt{a_1} = Y$, and so get

$$\bar{S} = X^2 + Y^2 + 2g_3X + 2f_3Y + c_1,$$

$$\bar{S}' = a_4(X^2 + Y^2) + 2g_4X + 2f_4Y + c_2.$$

The point to be made clear is that when we pass from S and S' to $\bar{S}$ and $\bar{S}'$ by an algebraical substitution of the form given in § 356 we are not entitled to divide out any factor in either $\bar{S}$ or $\bar{S}'$ unless indeed we divide it out of *both* of them. To this last there can be no objection.

Just in the same way if we have two conics S and S' which have double contact we cannot, because two conics with double contact can have their equations (§ 324) expressed

$$x^2 + y^2 + z^2 = 0, \qquad x^2 + y^2 + cz^2 = 0,$$

write $$S = x^2 + y^2 + z^2, \qquad S' = x^2 + y^2 + cz^2.$$

We may however write

$$S = x^2 + y^2 + z^2, \qquad S' = a(x^2 + y^2) + cz^2.$$

360. Illustrations of invariants.

To find the condition that the conics

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{.........(1)},$$

$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0 \quad \text{...(2)},$$

should be such that it is possible to inscribe a triangle in S' whose three sides shall touch S.

Let ABC be such a triangle. If we transform our co-ordinates so that ABC becomes the triangle of reference we shall have

$$S = \lambda^2 X^2 + \mu^2 Y^2 + \nu^2 Z^2 - 2\mu\nu YZ - 2\nu\lambda ZX - 2\lambda\mu XY,$$
$$S' = 2F'YZ + 2G'ZX + 2H'XY.$$

If now we write $X' = \lambda X$, $Y' = \mu Y$, $Z' = \nu Z$ we shall get

$$S = X'^2 + Y'^2 + Z'^2 - 2Y'Z' - 2Z'X' - 2X'Y' \quad \ldots(3),$$
$$S' = 2FY'Z' + 2GZ'X' + 2HX'Y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If now Δ, Δ', Θ, Θ' refer to (3) and (4), we have

$$\Delta = -4,$$
$$\Delta' = 2FGH,$$
$$\Theta = 4(F + G + H),$$
$$\Theta' = -F^2 - G^2 - H^2 + 2GH(-1) + 2HF(-1) + 2FG(-1)$$
$$= -(F + G + H)^2.$$

From these we derive the homogeneous relation, viz.

$$\Theta^2 = 4\Delta\Theta'.$$

As this is homogeneous it holds when Δ, Θ, Θ' refer to (1) and (2).

This then is the necessary condition if the conics S and S' are such that a triangle can be inscribed in S' so that its sides touch S, or, as we may put it, circumscribed to S and inscribed to S'.

361. We now proceed to enquire whether the condition $\Theta^2 = 4\Delta\Theta'$ is *sufficient* to ensure that S and S' are such that a triangle circumscribed to S will be inscribed to S'.

Let A and B be two points on S' such that AB is a tangent to S. Draw the other tangents from A and B to S, and let them meet in C.

Taking ABC for triangle of reference we may write

$$S \equiv x^2 + y^2 + z^2 - 2yz - 2zx - 2xy,$$
$$S' \equiv cz^2 + 2fyz + 2gzx + 2hxy.$$

Whence we have

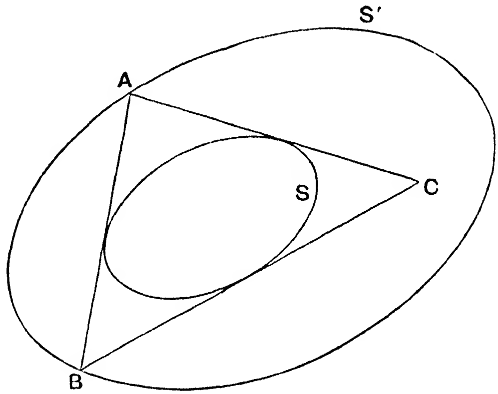
$$\Theta = 4(f+g+h), \quad \Delta = -4,$$

$$\Theta' = -(f+g+h)^2 + 2ch,$$

therefore the condition $\Theta^2 - 4\Delta\Theta' = 0$ becomes

$$16(f+g+h)^2 - 16(f+g+h)^2 + 32ch = 0,$$

$$\therefore\ c = 0, \text{ or } h = 0.$$



But if $h = 0$, S' becomes two straight lines

$$z(cz + 2fy + 2gx) = 0.$$

Excluding this case we see that

$$S' \equiv 2fyz + 2gzx + 2hxy,$$

that is C lies on S'.

Thus if S and S' be two non-degenerate conics satisfying the relation $\Theta^2 = 4\Delta\Theta'$, they are such that some triangle circumscribed to S is inscribed to S'. And we shall now shew that when there is one such triangle there is an infinite number.

362. Proposition. *If one triangle can be inscribed in a conic S' and circumscribed to another conic S, then an infinite number of such triangles can be drawn.*

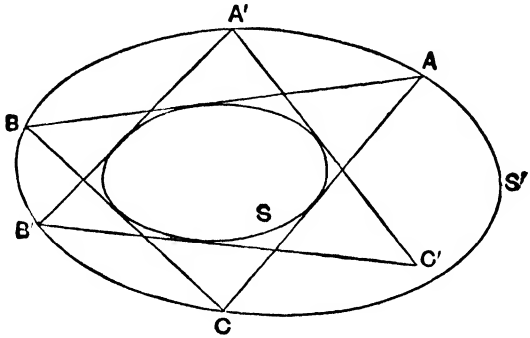
Let ABC be such a triangle.

Let A' be any other point on S'.

Draw $A'B'$ to touch S and to cut S' in B'.

Draw $B'C'$ and $A'C'$ to touch S and to meet in C'.

Then the six vertices A, B, C, A', B', C' all lie on a conic (*Pure Geometry*, § 252).



But the conic through the five points A, B, C, A', B' is S'.

Therefore C' must lie on S' also.

Thus the triangle $A'B'C'$ satisfies the conditions.

Thus we see that an infinite number of these triangles can be drawn if one can.

363. We proceed now to a further problem in which we shall make use of projections.

To find the condition that the two conics

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots\ldots(1),$$

$$S' \equiv a'x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0 \ldots\ldots(2),$$

should be such that the sides of a quadrilateral having its vertices on S' should touch S.

Let AB, BC, CD, DA be the sides touching S of the quadrilateral whose vertices A, B, C, D lie on S'.

Let PQR be the diagonal triangle for the *quadrangle* A, B, C, D. Then PQR is a self-polar triangle for S'.

Now project S' into a circle with P projected into the centre. Then QR goes to infinity, and $ABCD$ becomes a parallelogram, and being inscribed in a circle it must be a rectangle.

Thus, using small letters in the projection, a, b, c, d are points on the director circle of $\bar{S}$.

Thus $\bar{S}$ and $\bar{S}'$ are a conic and its director circle.

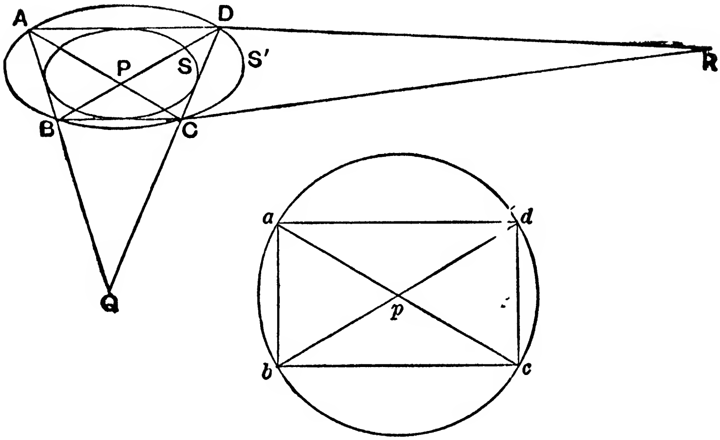
So then we may take

$$\bar{S} \equiv ax^2 + by^2 + c,$$

$$\bar{S}' \equiv a'(x^2 + y^2) + a'\left(\frac{c}{a} + \frac{c}{b}\right),$$

$$\therefore\ \frac{\bar{S}}{c} \equiv \frac{a}{c}x^2 + \frac{b}{c}y^2 + 1,$$

$$\frac{\bar{S}'}{c} \equiv \frac{a'}{c}(x^2 + y^2) + a'\left(\frac{1}{a} + \frac{1}{b}\right).$$



As our k equation arises from the condition that $\bar{S} + k\bar{S}'$ should be the product of two linear factors, we may omit the common factor $\frac{1}{c}$ and take

$$\bar{\bar{S}} \equiv \frac{a}{c}x^2 + \frac{b}{c}y^2 + 1,$$

$$\bar{\bar{S}}' \equiv \frac{a'}{c}(x^2 + y^2) + a'\left(\frac{1}{a} + \frac{1}{b}\right).$$

We now write $x\sqrt{a'} = \sqrt{c}\,X$, $y\sqrt{a'} = \sqrt{c}\,Y$, and get

$$\bar{\bar{S}} \equiv \frac{a}{a'}X^2 + \frac{b}{a'}Y^2 + 1 \equiv \alpha X^2 + \beta Y^2 + 1 \text{ (say)},$$

$$\bar{\bar{S}}' \equiv X^2 + Y^2 + \left(\frac{a'}{a} + \frac{a'}{b}\right) \equiv X^2 + Y^2 + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right).$$

Now form Δ, Δ', Θ, Θ' for these; we have

$$\Delta = \alpha\beta, \quad \Delta' = \frac{1}{\alpha} + \frac{1}{\beta},$$

$$\Theta = 2(\alpha + \beta), \quad \Theta' = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 3 = \frac{(\alpha+\beta)^2}{\alpha\beta} + 1.$$

These give $\Delta\Delta' = \alpha + \beta$, $\therefore\ \Theta = 2\Delta\Delta'$, and $\Theta' = \Delta\Delta'^2 + 1$,

$$\therefore\ \Theta\Theta' = 2\Delta\Delta' + 2\Delta^2\Delta'^3,$$

$$\therefore\ \Delta\Theta\Theta' = 2\Delta^2\Delta' + 2\Delta^3\Delta'^3,$$

$$\therefore\ 4\Delta\Theta\Theta' = 8\Delta^2\Delta' + \Theta^3,$$

$$\therefore\ \Theta^3 - 4\Delta\Theta\Theta' + 8\Delta^2\Delta' = 0.$$

This, being a homogeneous relation, holds when Δ, Δ', Θ, Θ' refer to the original conics S and S'.

The above relation is necessary that the conics (1) and (2) should be such that a quadrilateral circumscribed to (1) should be inscribed in (2). It must not however be assumed that it is *sufficient* (see Ex. 10 at end of chapter).

It is clear from the projection effected above that if *one* quadrilateral can be inscribed in S' and circumscribed to S, there is an infinite number of such quadrilaterals.

364. Condition for single contact.

To find the condition that the conics $S = 0$, $S' = 0$ should touch.

In general the conics cut in four points and we have three possible pairs of lines through their points of intersection, viz.

$$S + kS' = 0,$$

where k is given by

$$\Delta' k^3 + \Theta' k^2 + \Theta k + \Delta = 0 \quad\text{.................(1).}$$

If the conics touch, these three pairs of lines reduce to two. That is, this equation giving k has a pair of equal roots.

Now a double root of (1) is also a root of

$$3\Delta' k^2 + 2\Theta' k + \Theta = 0 \quad\text{.................(2).}$$

The condition that the conics should touch is then the result of eliminating k between (1) and (2).

This can be effected thus: (1) $\times 3 -$ (2) $\times k$ gives

$$\Theta' k^2 + 2\Theta k + 3\Delta = 0 \quad \dots\dots\dots\dots\dots\dots(3).$$

From (2) and (3) we have

$$\frac{k^2}{2(3\Delta\Theta' - \Theta^2)} = \frac{k}{\Theta\Theta' - 9\Delta\Delta'} = \frac{1}{2(3\Delta'\Theta - \Theta'^2)}.$$

Whence $\quad (\Theta\Theta' - 9\Delta\Delta')^2 = 4(3\Delta\Theta' - \Theta^2)(3\Delta'\Theta - \Theta'^2)$,
which becomes on multiplying out

$$\Theta^2\Theta'^2 + 18\Delta\Delta'\Theta\Theta' - 27\Delta^2\Delta'^2 - 4\Delta\Theta'^3 - 4\Delta'\Theta^3 = 0.$$

This condition is also *sufficient* that the conics should touch; for by the theory of equations we know that the expression on the left side is a multiple of the square of the product of the differences of the roots of the equation (1). If then it is zero, two roots of (1) must be equal.

365. Equation of common pair of chords of two conics which touch.

Suppose that A, B, C, D are the four points of intersection of two conics S and S'. The three pairs of common chords are

$$AC, BD;\ AD, BC;\ AB, CD.$$

Now if B coincide with A so that the conics touch at A, it is the first two of these pairs that become the same, thus the duplicate root k in the last paragraph will be that which gives the pair of chords AC, AD when the conics touch at A and cut in C and D.

But from the work of § 364, we see that this value of k is

$$\frac{\Theta\Theta' - 9\Delta\Delta'}{2(3\Delta'\Theta - \Theta'^2)}.$$

Hence the equation of the common pair of chords through the point of contact of two conics $S=0$, $S'=0$ which touch is

$$2(3\Delta'\Theta - \Theta'^2)S + (\Theta\Theta' - 9\Delta\Delta')S' = 0.$$

366. Double contact.

We naturally seek next to discover the condition that two conics S and S' should have double contact.

We know (§ 324) that we can reduce these so that

$$S \equiv a(x^2 + y^2) + cz^2,$$
$$S' \equiv x^2 + y^2 + z^2,$$

so that we have

$$\Delta = a^2c \ldots\ldots\ldots\ldots\ldots(1), \qquad \Delta' = 1 \ldots\ldots\ldots\ldots\ldots(2),$$
$$\Theta = 2ac + a^2 \ldots\ldots\ldots(3), \qquad \Theta' = 2a + c \ldots\ldots\ldots(4).$$

We should then have to eliminate c and a between (1), (3) and (4) and make our resulting relation homogeneous by means of (2). It will be found however that the result obtained is the same as that which we had for single contact. This relation then is necessary for double contact but not sufficient. There must be other conditions satisfied if the conics are to have double contact. These will be set forth in a subsequent chapter.

367. Condition for three point contact.

In this case three of the points A, B, C, D must coincide, say A, B, C; then we have only one pair of common chords, viz. the tangent at A, and AD.

Thus the equation giving k must have three roots equal, i.e.

$$k^3 + \frac{\Theta'}{\Delta'}k^2 + \frac{\Theta}{\Delta'}k + \frac{\Delta}{\Delta'} \equiv (k + \lambda)^3,$$

$$\therefore \quad 3\lambda = \frac{\Theta'}{\Delta'}, \quad 3\lambda^2 = \frac{\Theta}{\Delta'}, \quad \lambda^3 = \frac{\Delta}{\Delta'}.$$

Whence
$$\frac{\Theta'}{3\Delta'} = \frac{\Theta}{\Theta'} = \frac{3\Delta}{\Theta},$$

each of these being equal to λ.

368. Pair of tangents.

To find the equation of the pair of tangents to

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0,$$

the chord of contact being $lx + my + nz = 0$.

Here we take $\quad S' \equiv (lx + my + nz)^2,$

so that we have $\quad \Delta' = 0, \ \Theta' = 0.$

The pair of tangents is a pair of common chords of $S=0$, $S'=0$.

And the pairs of common chords are given by

$$S+kS'=0,$$

where $$\Delta' k^3+\Theta' k^2+\Theta k+\Delta=0.$$

Putting $\Delta'=0$, $\Theta'=0$ we have $k=-\dfrac{\Delta}{\Theta}$.

Thus the pair of tangents is

$$\Theta S=\Delta(lx+my+nz)^2,$$

where

$$\Theta=(bc-f^2)l^2+(ca-g^2)m^2+(ab-h^2)n^2+2(gh-af)mn \\ +2(hf-bg)nl+2(fg-ch)lm$$

$$=Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm.$$

Writing this Σ, we have as the equation of the pair of tangents

$$\Sigma S=\Delta(lx+my+nz)^2,$$

which is the same as

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix} S+\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}(lx+my+nz)^2=0.$$

As a particular case of the above, we see that the equation of the asymptotes of

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

in areal coordinates is

$$\begin{vmatrix} a, & h, & g, & 1 \\ h, & b, & f, & 1 \\ g, & f, & c, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} S+\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}(x+y+z)^2=0.$$

And the equation of the asymptotes of the conic represented by the general equation in any system of homogeneous coordinates is

$$\begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} S + \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} (\alpha x + \beta y + \gamma z)^2 = 0,$$

where $\alpha x + \beta y + \gamma z = 0$ is the equation of the line at infinity.

369. Invariants and reciprocation.

If the condition that a geometrical relation should exist between two conics S and S' is determined by a homogeneous function of

$$\Delta, \Theta, \Theta', \Delta'$$

equal to zero, the reciprocal relation is determined by the same function of

$$\Delta^2, \Delta\Theta', \Delta'\Theta, \Delta'^2$$

equal to zero.

For suppose the two conics S and S' reduced to

$$S \equiv ax^2 + by^2 + cz^2,$$

$$S' \equiv x^2 + y^2 + z^2.$$

Then $\Delta = abc$, $\Theta = bc + ca + ab$, $\Theta' = a + b + c$, $\Delta' = 1$.

If we reciprocate with respect to S', the conics become

$$S_1 \equiv bcx^2 + cay^2 + abz^2,$$

and $$S' \equiv x^2 + y^2 + z^2,$$

and

$$\Delta_1 = a^2b^2c^2, \ \Theta_1 = abc(a + b + c), \ \Theta_1' = bc + ca + ab, \ \Delta_1' = 1 = \Delta'^2$$
$$= \Delta^2, \qquad = \Delta\Theta', \qquad = \Theta\Delta'.$$

Now let $$\phi(\Delta_1, \Theta_1, \Theta_1', \Delta_1') = 0$$

be the homogeneous equation that must hold if a certain geometrical relation holds between S_1 and S'; then the reciprocal relation holds between S and S'; and the condition is equivalent to

$$\phi(\Delta^2, \Delta\Theta', \Delta'\Theta, \Delta'^2) = 0.$$

Thus the proposition is proved.

Illustration. The condition that triangles inscribed in S' should be circumscribed to S is as we have seen $\Theta^2 = 4\Delta\Theta'$,

hence the condition that triangles circumscribed to S' should be inscribed in S is

$$\Delta^2\Theta'^2 = 4\Delta^2\Delta'\Theta, \quad \text{that is} \quad \Theta'^2 = 4\Delta'\Theta,$$

which is obviously correct.

370. Geometrical interpretation of $\Theta = 0$.

It can be seen that $\Theta = 0$ for two non-degenerate conics S and S' in the two following cases:

(i) If some triangle self-conjugate to S is inscribed to S'; for then we can take

$$S = x^2 + y^2 + z^2,$$
$$S' = 2fyz + 2gzx + 2hxy.$$

(ii) If some triangle circumscribed to S is self-conjugate for S'; for then we can take

$$S = x^2 + y^2 + z^2 - 2yz - 2zx - 2xy,$$
$$S' = ax^2 + by^2 + cz^2.$$

We will now shew that if $\Theta = 0$ *both* the geometrical properties (i) and (ii) hold.

For let A be any point on S'. Let the polar of A with respect to S cut S' in B. Take C the pole of AB with respect to S. Then ABC is a self-conjugate triangle for S. Taking it for the triangle of reference, we may write

$$S = x^2 + y^2 + z^2,$$
$$S' = cz^2 + 2fyz + 2gzx + 2hxy,$$

since S' goes through A and B.

For these $\qquad \Theta = c, \quad \therefore\ c = 0,$

that is C lies on S'. Thus ABC, a triangle self-conjugate for S, is inscribed in S'.

Hence $\Theta = 0$ is the necessary and sufficient condition that some triangle self-conjugate for S should be inscribed in S'.

Reciprocating we have $\Delta\Theta' = 0$; i.e. $\Theta' = 0$ (for $\Delta \neq 0$) is the necessary and sufficient condition that a triangle self-conjugate for S should be circumscribed to S'.

That is $\Theta = 0$ is the necessary and sufficient condition that a triangle self-conjugate for S' should be circumscribed to S.

Thus if $\Theta = 0$ both the geometrical properties (i) and (ii) hold simultaneously.

It is further clear that there is an infinite number of triangles satisfying the conditions.

EXAMPLES.

1. Two circles S and S' of radii r and r' and with centres at distance d apart are such that the sides of a triangle inscribed in S' touch S, prove $d^2 = r'^2 \pm 2rr'$.

[Take centre of S for origin and line joining the centres for x-axis so that the circles are

$$S \equiv x^2 + y^2 - r^2 = 0,$$

$$S' \equiv (x - d)^2 + y^2 - r'^2 = 0,$$

that is

$$S \equiv x^2 + y^2 - r^2z^2,$$

$$S \equiv x^2 + y^2 + (d^2 - r'^2)\, z^2 - 2dzx.$$

Then use

$$\Theta^2 - 4\Delta\Theta' = 0.]$$

2. If $S = 0$ be a non-degenerate conic and S' a pair of straight lines, then :

(i) If $\Theta' = 0$ the point of intersection of the lines S' lies on S and conversely.

(ii) If $\Theta = 0$ the lines S' are conjugate lines for S and conversely.

[Take $S \equiv (a, b, c, f, g, h)(x, y, z)^2$ and $S' \equiv 2yz$.]

3. Prove that if $\Theta = 0$ for the conics S and S', then S and S' can be projected into a circle and a rectangular hyperbola, the latter passing through the centre of the circle.

Prove also that S and S' can be projected into a parabola and a circle, the centre of which lies on the directrix of the parabola.

4. Deduce from § 370 that the circle circumscribing a triangle self-conjugate for a conic S cuts the director circle of the conic at right angles.

[Take the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $(x-h)^2+(y-k)^2=r^2$ and express that $\Theta=0$.]

5. Prove that $\Theta=0$ if S be a hyperbola and S' the director circle of its conjugate.

6. Prove that if S' is a given conic, the locus of the centre of a variable conic S such that (i) $\Theta=0$ and (ii) a given triangle is self-conjugate for S, is a straight line.

7. The condition that it should be possible to inscribe triangles in the ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$ whose sides touch $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

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$$\pm\frac{a}{a'}\pm\frac{b}{b'}=1.$$

53. 8. If the tangents to S at two of its points of intersection with S' intersect on S', then

$$\Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta'=0.$$

9. If the line joining the points of contact with the conic S' of two common tangents to S and S' touches S, prove that

$$\Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta'=0.$$

Shew how this result could have been inferred from Example 8.

10. If S and S' be two conics which have a common self-conjugate triangle and if

$$\Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta'=0,$$

then *either* (i) quadrilaterals can be circumscribed to S which are inscribed to S', *or* (ii) the tangents to S at two of its points of intersection with S' intersect on S'.

[The conics have a common self-conjugate triangle ABC. Project S' into a circle and BC to infinity. We shall then have a conic $\bar{S}$ and a circle $\bar{S}'$ concentric with it. If r be the radius of the circle and α^2, β^2 the squares of the semi-axes of the conic, it will be found that the condition leads to either $r^2=\alpha^2+\beta^2$ or $r^2=\pm(\alpha^2-\beta^2)$. The first gives (i) and the second (ii).]

11. If two conics S and S' have double contact and if in addition an n-sided polygon can be drawn whose sides touch one of the conics and whose vertices lie on the other, then

$$\Theta\Theta' = \Delta\Delta'\left(1 + 2\cos^2\frac{r\pi}{n}\right)\left(1 + 2\sec^2\frac{r\pi}{n}\right),$$

where r is an integer prime to n.

Prove that if one such polygon can be drawn, there is an infinite number that can be drawn.

[Conics with double contact can be projected into concentric circles by projecting the points of contact into the circular points (*Pure Geometry*, § 134).]

12. Prove that the same relation holds as in **Ex. 10** when the conics have *single* contact.

13. If two parabolas with the same focus are inscribed and circumscribed respectively to a triangle, prove that the angle between their axes is $2\cos^{-1}\frac{1}{2}\sqrt{\frac{L}{l}}$ where L and l are their latera recta.

14. Shew that the ratio of the curvatures at the point of contact of the conics $S = 0$, $S' = 0$ which touch, is the ratio of the two unequal roots of the equation

$$\Delta' k^3 + \Theta' k^2 + \Theta k + \Delta = 0.$$

[Refer the conics to their common tangent and normal as axes (§ 237, Ex. 2).]

15. If A, B, C, D are the points of intersection of the conics $S = 0$, $S' = 0$ and P be any point on $S = 0$, the cross ratio λ of the pencil $P(ABCD)$ is given by

$$\frac{(\lambda^2 - \lambda + 1)^3}{(\lambda + 1)^2(2\lambda^2 - 5\lambda + 2)^2} = \frac{\{\Theta^2 - 3\Delta\Theta'\}^3}{\{2\Theta^3 - 9\Delta(\Theta\Theta' - 3\Delta\Delta')\}^2}.$$

[Take $S \equiv ax^2 + by^2 + cz^2$, and $S' \equiv x^2 + y^2 + z^2$, and see § 338.]

16. The condition that the pencil formed by joining any point on the conic S with its four points of intersection with S' may be harmonic is

$$27\Delta^2\Delta' - 9\Delta\Theta\Theta' + 2\Theta^3 = 0.$$

17. The conic $S+\lambda S'=0$ degenerates into a pair of lines for the values λ_1, λ_2, λ_3 of λ; prove that the cross ratio of the pencil joining any point on $S+\lambda S'=0$ to the four common points of the system is

$$\frac{(\lambda-\lambda_1)(\lambda_2-\lambda_3)}{(\lambda-\lambda_3)(\lambda_2-\lambda_1)}.$$

Shew also that for the pair of conics $S+\mu_2 S'=0$, $S+\mu_1 S'=0$, the condition $\Theta=0$ becomes

$$\frac{\lambda_1-\mu_1}{\mu_2-\lambda_1}+\frac{\lambda_2-\mu_1}{\mu_2-\lambda_2}+\frac{\lambda_3-\mu_1}{\mu_2-\lambda_3}=0.$$

18. *ABCDEF* is a hexagon inscribed in a conic S' so that any two consecutive vertices are conjugate points with respect to another conic S, prove that

$$\Theta^3-4\Delta(\Theta\Theta'-2\Delta\Delta')=0.$$

19. If S and S' be two conics such that a pair of their common chords are conjugate lines with regard to S, then

$$2\Theta^3-9\Theta\Theta'\Delta+27\Delta^2\Delta'=0.$$

20. If S and S' be two conics such that a pair of their common chords are conjugate lines for both conics, then $\Theta^3\Delta'=\Theta'^3\Delta$ and either $\Theta\Theta'=9\Delta\Delta'$.

21. If the four points of contact with $S=0$ of the common tangents to $S=0$, $S'=0$ be joined to any point of S and the lines so formed determine a harmonic pencil, shew that

$$2\Theta'^3-9\Theta\Theta'\Delta'+27\Delta\Delta'^2=0.$$

CHAPTER XVIII.

TANGENTIAL EQUATIONS.—ENVELOPES.

371. Tangential equations defined.

We have seen that the condition that the line

$$lx + my + nz = 0 \quad \text{......................(1)}$$

should touch the conic

$$(a, b, c, f, g, h)(x, y, z)^2 = 0 \quad \text{..............(2)}$$

is

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0,$$

which we write

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \quad \text{...(3)},$$

A, B, C, F, G, H being the minors with their proper signs of a, b, c, f, g, h in the determinant

$$\Delta \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

The relation (3) giving the condition that the line (1) should touch (2) is called the *tangential equation* of the conic (2).

372. Reciprocal relation between point and tangential equations.

We now proceed to shew that when the tangential equation of a conic is given, the ordinary equation of the conic, which we shall call the point equation, can be found from it exactly as the tangential equation is formed from the point equation.

That is to say:

If $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$

be the tangential equation, then

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

is the point equation.

For let the point equation be

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0.$$

Therefore the tangential equation is

$$(b'c' - f'^2)\, l^2 + (c'a' - g'^2)\, m^2 + (a'b' - h'^2)\, n^2 + 2\,(g'h' - a'f')\, mn \\ + 2\,(h'f' - b'g')\, nl + 2\,(f'g' - c'h')\, lm = 0.$$

Therefore we must have

$$\frac{a}{b'c' - f'^2} = \frac{b}{c'a' - g'^2} = \frac{c}{a'b' - h'^2} = \frac{f}{g'h' - a'f'} \\ = \frac{g}{h'f' - b'g'} = \frac{h}{f'g' - c'h'} = \lambda \text{ (say)},$$

$$\therefore\ bc - f^2 = \lambda^2 \{(c'a' - g'^2)(a'b' - h'^2) - (g'h' - a'f')^2\} \\ = \lambda^2 a' \{a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2\} = \lambda^2 a'\Delta'.$$

So $\qquad ca - g^2 = \lambda^2 b'\Delta'$ and $ab - h^2 = \lambda^2 c'\Delta'$.

Also $\quad gh - af = \lambda^2 \{(h'f' - b'g')(f'g' - c'h')$

$$- (b'c' - f'^2)(g'h' - a'f')\} = \lambda^2 f'\Delta'.$$

So $\qquad hf - bg = \lambda^2 g'\Delta'$ and $fg - ch = \lambda^2 h'\Delta'$.

Wherefore the point equation of the conic is

$$(bc - f^2)\, x^2 + (ca - g^2)\, y^2 + (ab - h^2)\, z^2 + 2\,(gh - af)\, yz \\ + 2\,(hf - bg)\, zx + 2\,(fg - ch)\, xy = 0,$$

that is $\quad Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$

373. Connection with reciprocation.

This relation of reciprocity between the point equation and the tangential equation of a conic is easily seen to be connected with that of polar reciprocation.

For if $\qquad (a, b, c, f, g, h)(x, y, z)^2 = 0$ ....(1)

be the point equation, then

$$(A, B, C, F, G, H)(l, m, n)^2 = 0 \quad \ldots\ldots\ldots\ldots(2)$$

is the tangential equation.

But $$(A, B, C, F, G, H)(x, y, z)^2 = 0 \quad \ldots\ldots\ldots\ldots(3)$$

is (§ 318) the polar reciprocal of (1) with respect to the conic

$$x^2 + y^2 + z^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

And, as we have seen, (1) is the polar reciprocal of (3) with respect to (4).

Thus if $$(a, b, c, f, g, h)(l, m, n)^2 = 0$$

be the tangential equation of a conic, the conic is the polar reciprocal with respect to (4) of

$$(a, b, c, f, g, h)(x, y, z)^2 = 0,$$

that is, its point equation is

$$(A, B, C, F, G, H)(x, y, z)^2 = 0.$$

We observe too that equation (2) expresses the fact that the point (l, m, n) lies on (3).

Also the point (l, m, n) is the pole of the line

$$lx + my + nz = 0$$

with respect to (4).

Hence the tangential equation of a conic merely expresses the fact that the pole with respect to (4) of a line touching the conic lies on the polar reciprocal of the conic with respect to (4).

Example. Find the point equations corresponding to the tangential equations

(1) $al^2 + bm^2 + cn^2 = 0.$

(2) $\dfrac{\lambda}{l} + \dfrac{\mu}{m} + \dfrac{\nu}{n} = 0.$

374. Tangential equation of the first degree.

The general tangential equation of the first degree is

$$Al + Bm + Cn = 0.$$

This tells us nothing about the line $lx + my + nz = 0$ except that it passes through the point (A, B, C). The tangential equation of the first degree then represents a point. As we

have seen, the tangential equation of the second degree represents a conic. There is however a special case where it represents two points. This is when it reduces to the form

$$(Al + Bm + Cn)(A'l + B'm + C'n) = 0.$$

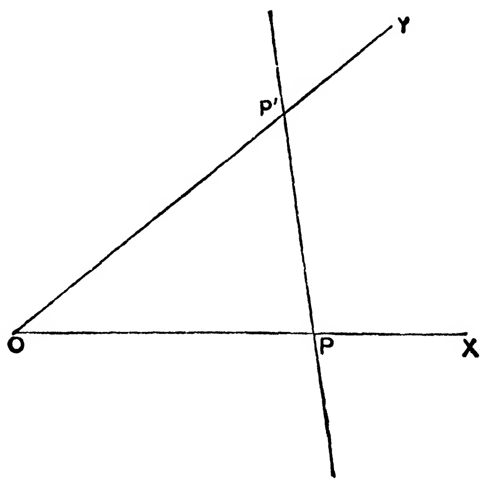
For this equation is only satisfied if $Al + Bm + Cn = 0$ or $A'l + B'm + C'n = 0$.

375. Envelope of lines joining corresponding points of two homographic ranges.

We will now prove the proposition which we stated without proof at the end of § 352.

The lines joining corresponding points of two homographic ranges on two intercepting lines, the point of intersection of the lines not corresponding to itself in the two ranges, all touch a conic, which touches the two lines of the ranges.

Let OX, OY be the lines of the ranges. These we will take for axes of coordinates.



Let P and P' be corresponding points in the two ranges.

Let $$OP = p, \quad OP' = p'.$$

Let $$lx + my + nz = 0,$$
which is the same as $$lx + my + n = 0,$$
be the equation of PP'.

$$\therefore\ p = -\frac{n}{l},\ \ p' = -\frac{n}{m}.$$

But $$App' + Hp + H'p' + B = 0,$$
where A, B, H and H' are constants.

Therefore $$A\frac{n^2}{lm} - \frac{Hn}{l} - \frac{H'n}{m} + B = 0,$$

$$\therefore\ An^2 - Hmn - H'nl + Blm = 0 \ldots\ldots\ldots\ldots (1).$$

Therefore the line $lx + my + nz = 0$ touches a conic whose tangential equation is (1).

Moreover as (1) is satisfied for the values (1, 0, 0), (0, 1, 0) of (l, m, n) it is clear that the conic touches

$$x = 0 \text{ and } y = 0,$$

that is the two lines of the ranges.

Thus the proposition is proved. Another proof will be given in § 377.

In the special case where $A = 0$ the conic also touches the line $z = 0$, that is the line at infinity. In this case, and in this case only, the conic is a parabola.

376. Envelopes.

The line $$lx + my + nz = 0,$$
when the relation

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$$

holds, touches (or 'envelops') the conic

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

The conic then is called the *envelope* of the line.

The coefficients l, m, n in the equation of the line are called *the coordinates of the line* (§ 269). If then the coordinates of a line are connected by a homogeneous relation of the second order the envelope of the line is a conic.

Suppose now that we have the equation of a line in the form

$$(l_1x + m_1y + n_1z) + k(l_2x + m_2y + n_2z) + k^2(l_3x + m_3y + n_3z) = 0,$$

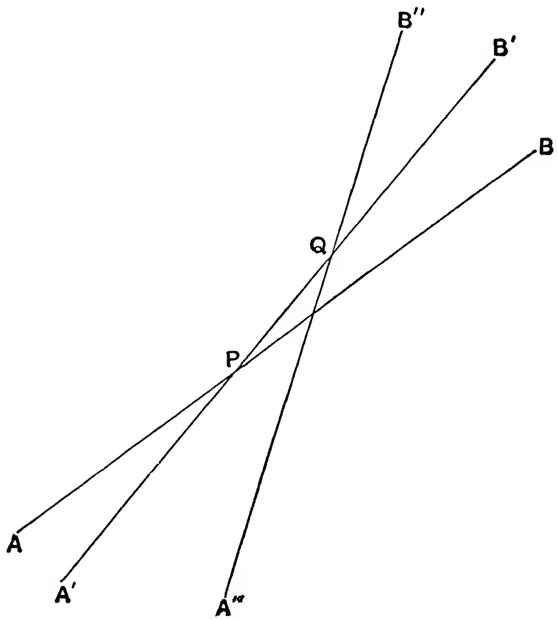
and we want the envelope of this line due to changes in k, the l's, m's and n's not changing with k.

Now suppose that AB is the position of this line for some particular value of k which we will denote by k_1.

And let $A'B'$ be the consecutive position of the line, differing but slightly from AB, for a particular value of k differing but slightly from k_1. We will denote this particular value by $k_1 + \delta k_1$ where δk_1 is very small.

Let AB and $A'B'$ intersect in P.

Let $A''B''$ be the consecutive position of the line related to $A'B'$ as $A'B'$ was to AB, and let $A''B''$ cut $A'B'$ in Q.



Then the locus of these points P, Q for the various positions of the line will be the envelope.

For as we see, P and Q being near points common both to $A'B'$ and the locus of P, the line $A'B'$ must touch the locus of P.

Similarly all the other lines must touch the locus of P. The locus of P gives then the envelope required.

This is expressed by saying that the envelope of the line is the locus of the ultimate intersections of consecutive lines of the system.

We will now write the equation of the line for short

$$L + kM + k^2N = 0.$$

The equation of the line AB is then

$$L + k_1M + k_1^2N = 0$$

and of $A'B'$ $\quad L + (k_1 + \delta k_1)M + (k_1 + \delta k_1)^2N = 0.$

For points common to these we have

$$\left.\begin{array}{r} L + k_1M + k_1^2N = 0 \\ \delta k_1(M + 2Nk_1) + (\delta k_1)^2 N = 0 \end{array}\right\},$$

that is

$$\left.\begin{array}{r} L + k_1M + k_1^2N = 0 \\ (M + 2Nk_1) + \delta k_1 N = 0 \end{array}\right\}.$$

Now δk_1 is very small, and becomes smaller and smaller the more $A'B'$ approximates to AB.

The locus of ultimate intersection of consecutive lines then is given by

$$\left.\begin{array}{r} L + k_1M + k_1^2N = 0 \\ M + 2Nk_1 = 0 \end{array}\right\},$$

that is

$$L + M\left(-\frac{M}{2N}\right) + \frac{M^2}{4N^2}N = 0,$$

that is

$$M^2 = 4LN,$$

or $\quad (l_2x + m_2y + n_2z)^2 = 4(l_1x + m_1y + n_1z)(l_3x + m_3y + n_3z),$

which is a conic.

It will be observed that the envelope is the condition that

$$L + Mk + Nk^2 = 0$$

should have equal roots in k. This fact is easy to remember and enables us quickly to write down the envelope of a line of this form.

But it will be observed that the above argument holds exactly the same for a *curve* whose equation can be expressed in the form $P + kQ + k^2R = 0$.

The envelope of such curves for variations of k is

$$Q^2 = 4PR.$$

That is to say this curve touches all the curves belonging to the family of curves $P + kQ + k^2R = 0$.

It will be seen from this paragraph that if

$$X^2 = \lambda YZ$$

be the equation of a conic, then the lines

$$\lambda Y + 2kX + k^2Z = 0,$$

for the different values of k are all tangents to the conic.

It can be proved in exactly the same way that the envelope of the curves whose equation is

$$P + Qk + Rk^2 + Sk^3 = 0,$$

for variations in k is the result of eliminating k between

$$\left.\begin{aligned} P + Qk + Rk^2 + Sk^3 &= 0 \\ Q + 2Rk + 3Sk^2 &= 0 \end{aligned}\right\}.$$

And the envelope of the curves whose equation is

$$P + Qk + Rk^2 + Sk^3 + Tk^4 = 0$$

is the result of eliminating k between this and

$$Q + 2Rk + 3Sk^2 + 4Tk^3 = 0.$$

But all these are but special cases of a general theorem of the Differential Calculus that the envelope of curves $\phi(k) = 0$ for variations of k is the result of eliminating k between

$$\left.\begin{aligned} \phi(k) &= 0 \\ \frac{d}{dk}\phi(k) &= 0 \end{aligned}\right\}.$$

By the same reasoning as that of § 375, we can see that if

$$P + Qk + Rk^2 = 0$$

be the *tangential* equation of a curve the tangential equation of the envelope is $Q^2 = 4PR$.

377. Prop. *The envelope of the line $lx + my + nz = 0$ where*

$$l : m : n = at^2 + bt + c : a't^2 + b't + c' : a''t^2 + b''t + c''$$

(a, b, c, etc. being constants) is a conic.

This proposition is the reciprocal of that of § 321. By that article the locus of (l, m, n) when l, m, n are connected by the given relation is a conic; and as the point (l, m, n) reciprocates into the line $lx + my + nz = 0$ with respect to the conic $x^2 + y^2 + z^2 = 0$, the proposition follows at once.

We can make use of this proposition to prove the property given in § 375. For if P and P' be corresponding points in the two homographic ranges the lines of which intersect in O, then if

$$OP = t, \quad OP' = \frac{at + b}{ct + d}$$

where a, b, c, d are constants.

Thus the equation of PP' is

$$\frac{x}{t} + \frac{y(ct + d)}{at + b} = 1,$$

that is $$(at + b)x + (ct^2 + dt)y - (at^2 + bt)z = 0.$$

Therefore the envelope of PP' is a conic.

378. Conics touching the common tangent of two given conics.

Suppose now we have two conics whose tangential equations are

$$\Sigma \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \text{ ......(1)},$$

$$\Sigma' \equiv a'l^2 + b'm^2 + c'n^2 + 2f'mn + 2g'nl + 2h'lm = 0 \text{...(2)}.$$

We seek to interpret the tangential equation

$$\Sigma + k\Sigma' = 0 \text{ ....(3)}.$$

It is the equation of a conic, for it is homogeneous of the second degree in l, m, n.

Moreover we can see that (3) touches all tangents which (1) and (2) have in common.

For if $lx + my + nz = 0$ be a tangent to (1) and (2) they are simultaneously satisfied, and therefore (3) is satisfied too.

Thus as, if $S = 0$, $S' = 0$ are the equations of two conics, $S + kS' = 0$ is the equation of a conic passing through their common points of intersection, so if $\Sigma = 0$, $\Sigma' = 0$ are the

tangential equations of two conics, $\Sigma + k\Sigma' = 0$ is the tangential equation of a conic touching their common tangents.

Consider now some special cases.

The equation

$$\Sigma + k(Ll + Mm + Nn)(L'l + M'm + N'n) = 0,$$

where L, M, N, L', M', N' are constant, is the tangential equation of a conic touching the common tangents of $\Sigma = 0$, and

$$(Ll + Mm + Nn)(L'l + M'm + N'n) = 0,$$

and thus it is the tangential equation of a conic touching the common tangents of $\Sigma = 0$, and $Ll + Mm + Nn = 0$ and also the common tangents of $\Sigma = 0$, and $L'l + M'm + N'n = 0$. That is to say it is the tangential equation of a conic touching the two pairs of tangents to the conic $S = 0$ drawn from the points (L, M, N) and (L', M', N').

In a similar way we can see that the equation

$$(Ll + Mm + Nn)(L'l + M'm + N'n) = (L''l + M''m + N''n)(L'''l + M'''m + N'''n)$$

is the tangential equation of a conic touching the lines joining (L, M, N) to (L'', M'', N'') and to (L''', M''', N''') and the lines joining (L', M', N') to (L'', M'', N'') and to (L''', M''', N''').

379. Equation of points of intersection of two conics.

To find the tangential equation of the four points of intersection of the conics

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0,$$

that is, to find the condition that the line $lx + my + nz = 0$ should pass through one of these four points.

The tangential equations of these conics are

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$
$$\Sigma' \equiv A'l^2 + B'm^2 + C'n^2 + 2F'mn + 2G'nl + 2H'lm = 0$$

And the equation of any conic through their points of intersection is

$$S + kS' = 0,$$

whose tangential equation is

$$\begin{vmatrix} a+ka', & h+kh', & g+kg', & l \\ h+kh', & b+kb', & f+kf', & m \\ g+kg', & f+kf', & c+kc', & n \\ l, & m, & n, & 0 \end{vmatrix} = 0.$$

This on being multiplied out gives

$$\Sigma + k\Phi + k^2\Sigma' = 0,$$

where

$$\begin{aligned}\Phi \equiv (bc'+b'c-2ff')\,l^2 &+ (ca'+c'a-2gg')\,m^2 \\ &+ (ab'+a'b-2hh')\,n^2 + 2\,(gh'+g'h-af'-a'f)\,mn \\ &+ 2\,(hf'+h'f-bg'-b'g)\,nl + 2\,(fg'+f'g-ch'-c'h)\,lm.\end{aligned}$$

Thus the tangential equation of the envelope of the system of conics through the four points of intersection of S and S' is

$$\Phi^2 = 4\Sigma\Sigma'.$$

But the envelope or locus of ultimate intersection of the consecutive curves of the family $S + kS' = 0$ is the four points of intersection of S and S', for no two conics can cut in more than four points, and all the curves of the family $S + kS'$ already pass through the same four points.

$$\therefore\ \Phi^2 = 4\Sigma\Sigma'$$

is the tangential equation of the four points of intersection of S and S'.

380. The student will have understood that $\Phi^2 - 4\Sigma\Sigma'$ in the last article must be the product of four linear factors in l, m, n of the form

$$(A_1l+B_1m+C_1n)\,(A_2l+B_2m+C_2n)\,(A_3l+B_3m+C_3n)\,(A_4l+B_4m+C_4n)$$

and then $(A_1B_1C_1)\,(A_2B_2C_2)\,(A_3B_3C_3)\,(A_4B_4C_4)$ give the coordinates of the four points of intersection.

Let us illustrate this by taking the conics

$$S \equiv ax^2 + by^2 - 1 = 0,$$
$$S' \equiv a'x^2 + b'y^2 - 1 = 0.$$

By actual solving we find that these intersect in the points

$$\left(\pm\sqrt{\frac{b'-b}{ab'-a'b}},\ \pm\sqrt{\frac{a-a'}{ab'-a'b}}\right).$$

Now let us verify that the same result is obtained from $\Phi^2-4\Sigma\Sigma'=0$.

We have $\Sigma=-bl^2-am^2+abn^2$,

$\Sigma'=-b'l^2-a'm^2-a'b'n^2$,

and $\Phi=-(b+b')l^2-(a+a')m^2+(ab'+a'b)n^2$.

$$\begin{aligned}\therefore\ \Phi^2-4\Sigma\Sigma'=&\ l^4\{(b+b')^2-4bb'\}+m^4\{(a+a')^2-4aa'\}\\&+n^4\{(ab'+a'b)^2-4aa'bb'\}\\&-2m^2n^2\{(a+a')(ab'+a'b)-4aa'b-4aa'b'\}\\&-2n^2l^2\{(b+b')(ab'+a'b)-4a'bb'-4abb'\}\\&+2l^2m^2\{(a+a')(b+b')-4a'b-4ab'\}\end{aligned}$$

$$\begin{aligned}=&(b-b')^2l^4+(a-a')^2m^4+(ab'-a'b)n^4-2(a-a')(ab'-a'b)m^2n^2\\&-2(b'-b)(ab'-a'b)n^2l^2-2(a-a')(b'-b)l^2m^2\end{aligned}$$

$$\begin{aligned}=&(l\sqrt{b'-b}+m\sqrt{a-a'}+n\sqrt{ab'-a'b})(-l\sqrt{b'-b}+m\sqrt{a-a'}+n\sqrt{ab'-a'b})\\&\times(l\sqrt{b'-b}-m\sqrt{a-a'}+n\sqrt{ab'-a'b})(l\sqrt{b'-b}+m\sqrt{a-a'}-n\sqrt{ab'-a'b}).\end{aligned}$$

Thus the four points of intersection have their coordinates proportional to

$$(\sqrt{b'-b},\ \sqrt{a-a'},\ \sqrt{ab'-a'b}),\ (-\sqrt{b'-b},\ \sqrt{a-a'},\ \sqrt{ab'-a'b}),$$
$$(\sqrt{b'-b},\ -\sqrt{a-a'},\ \sqrt{ab'-a'b},\ (\sqrt{b'-b}),\ \sqrt{a-a'},\ -\sqrt{ab'-a'b}).$$

But as the z coordinate is unity, the actual x and y coordinates are

$$\left(\frac{\sqrt{b'-b}}{\sqrt{ab'-a'b}},\ \frac{\sqrt{a-a'}}{\sqrt{ab'-a'b}}\right),\ \left(-\frac{\sqrt{b'-b}}{\sqrt{ab'-a'b}},\ \frac{\sqrt{a-a'}}{\sqrt{ab'-a'b}}\right),$$
$$\left(\frac{\sqrt{b'-b}}{\sqrt{ab'-a'b}},\ -\frac{\sqrt{a-a'}}{\sqrt{ab'-a'b}}\right),\ \left(-\frac{\sqrt{b-b'}}{\sqrt{ab'-a'b}},\ -\frac{\sqrt{a-a'}}{\sqrt{ab'-a'b}}\right).$$

These agree with what we get by actual solving of the equations.

381. Equation of four common tangents of two conics.

We pass now to the reciprocal of the problem of § 379, viz.

To find the equation of the four common tangents of two given conics

$$S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$
$$S'\equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0.$$

The tangential equations of these are

$$\Sigma\equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,$$
$$\Sigma'\equiv A'l^2+B'm^2+C'n^2+2F'mn+2G'nl+2H'lm=0.$$

Thus the tangential equation of a conic having as tangents the common tangents of the given conics is

$$\Sigma + k\Sigma' = 0.$$

And the point equation of this conic is (§ 372)

$$\begin{vmatrix} A+kA', & H+kH', & G+kG', & x \\ H+kH', & B+kB', & F+kF', & y \\ G+kG', & F+kF', & C+kC', & z \\ x, & y, & z, & 0 \end{vmatrix} = 0,$$

which is

$$\begin{aligned}(BC-F^2)x^2 &+ (CA-G^2)y^2 + (AB-H^2)z^2 + 2(GH-AF)yz \\ &+ 2(HF-BG)zx + 2(FG-CH)xy + k\mathbf{F} + k^2[(B'C'-F'^2)x^2 \\ &+ (C'A'-G'^2)y^2 + (A'B'-H'^2)z^2 + 2(G'H'-A'F')yz \\ &+ 2(H'F'-B'G')zx + 2(F'G'-C'H')xy] = 0,\end{aligned}$$

where

$$\begin{aligned}\mathbf{F} \equiv (BC'+B'C-2FF')x^2 &+ (CA'+C'A-2GG')y^2 \\ + (AB'+A'B-2HH')z^2 &+ 2(GH'+G'H-AF'-A'F)yz \\ &+ 2(HF'+H'F-BG'-B'G)zx \\ &+ 2(FG'+F'G-CH'-C'H)xy.\end{aligned}$$

Now $BC-F^2 = (ca-g^2)(ab-h^2)-(gh-af)^2 = a\Delta.$

So $CA-G^2 = b\Delta$ and $AB-H^2 = c\Delta.$

Also

$$GH-AF = (hf-bg)(fg-ch)-(bc-f^2)(gh-af) = f\Delta,$$

and similarly

$$HF-BG = g\Delta, \text{ and } FG-CH = h\Delta.$$

Thus the above equation becomes

$$\Delta S + k\mathbf{F} + k^2\Delta' S' = 0.$$

Hence $\mathbf{F}^2 = 4\Delta\Delta' SS'$ is the envelope of the system of conics touching the common tangents of the two given conics.

But the envelope is the four common tangents themselves.

Therefore, $\mathbf{F}^2 = 4\Delta\Delta' SS'$ is the equation of the four common tangents of $S=0$ and $S'=0$.

382. Reciprocal relation.

Let us notice that we could have deduced the equation of the four common tangents of

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0 \ldots\ldots\ldots\ldots\ldots(1),$$

$$S' \equiv (a', b', c', f', g', h')(x, y, z)^2 = 0 \ldots\ldots\ldots(2),$$

from the result of § 379.

For if we reciprocate these two conics with respect to

$$x^2 + y^2 + z^2 = 0,$$

we get
$$(A, B, C, F, G, H)(x, y, z)^2 = 0 \ldots\ldots\ldots\ldots\ldots(3),$$

$$(A', B', C', F', G', H')(x, y, z)^2 = 0 \ldots\ldots\ldots(4).$$

The tangential equation of the four common points of these two is, by § 379,

$$\Phi_1^2 = 4\Sigma_1\Sigma_1' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where Φ_1 is the same function of the capital letters $A, B, C\ldots A', B', C'$ etc., as Φ was of $a, b, c\ldots a', b', c'\ldots$, and

$$\Sigma_1 = A_1l^2 + B_1m^2 + C_1n^2 + 2F_1mn + 2G_1nl + 2H_1lm,$$

where A_1, B_1, C_1, etc. are the minors with their proper signs of A, B, C, etc. in

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix},$$

that is $A_1 = a\Delta$, $B_1 = b\Delta$, $C_1 = c\Delta$, $F_1 = f\Delta$ and so on, where

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

that is
$$\Sigma_1 = \Delta (a, b, c, f, g, h)(l, m, n)^2$$

and Σ_2 is similarly $\Delta' (a', b', c', f', g', h')(l, m, n)$.

Now if in the equation (5) we write x, y, z for l, m, n respectively we shall obtain the reciprocal with respect to $x^2 + y^2 + z^2 = 0$ of the four common points of (3) and (4), that is we shall obtain the four common tangents of (1) and (2), viz.

$$\mathbf{F}^2 = 4\Delta\Delta' SS',$$

where F stands for what Φ of § 379 becomes when we change l, m, n to x, y, z respectively and write capital letters in the coefficients.

This result agrees with what we have obtained independently in § 381.

383. Circular points at infinity in generalised coordinates.

To find the tangential equation of the circular points at infinity in general homogeneous coordinates.

Here we want to find the condition that the line

$$lx + my + nz = 0 \quad \text{.....................(1)}$$

should pass through one or other of the two circular points.

The circular points are given by

$$\alpha x + \beta y + \gamma z = 0 \quad \text{.....................(2)},$$

$$\frac{a^2}{\alpha} yz + \frac{b^2}{\beta} zx + \frac{c^2}{\gamma} xy = 0 \quad \text{..................(3)},$$

for these give the intersection of the line at infinity with the circumcircle of the triangle of reference.

Treating these three equations as simultaneous, we have from (1) and (2)

$$\frac{\alpha x}{\frac{m}{\beta} - \frac{n}{\gamma}} = \frac{\beta y}{\frac{n}{\gamma} - \frac{l}{\alpha}} = \frac{\gamma z}{\frac{l}{\alpha} - \frac{m}{\beta}}.$$

Substituting in (3) we get

$$\frac{a^2}{\alpha\beta\gamma}\left(\frac{n}{\gamma} - \frac{l}{\alpha}\right)\left(\frac{l}{\alpha} - \frac{m}{\beta}\right) + \frac{b^2}{\alpha\beta\gamma}\left(\frac{l}{\alpha} - \frac{m}{\beta}\right)\left(\frac{m}{\beta} - \frac{n}{\gamma}\right)$$
$$+ \frac{c^2}{\alpha\beta\gamma}\left(\frac{m}{\beta} - \frac{n}{\gamma}\right)\left(\frac{n}{\gamma} - \frac{l}{\alpha}\right) = 0,$$

that is

$$a^2\frac{l^2}{\alpha^2} + b^2\frac{m^2}{\beta^2} + c^2\frac{n^2}{\gamma^2} - (b^2 + c^2 - a^2)\frac{mn}{\beta\gamma} - (c^2 + a^2 - b^2)\frac{nl}{\gamma\alpha}$$
$$- (a^2 + b^2 - c^2)\frac{lm}{\alpha\beta} = 0$$

which we may write

$$\Omega \equiv \frac{a^2}{\alpha^2} l^2 + \frac{b^2}{\beta^2} m^2 + \frac{c^2}{\gamma^2} n^2 - \frac{2bc}{\beta\gamma} mn \cos A$$
$$- \frac{2ca}{\gamma\alpha} nl \cos B - \frac{2ab}{\alpha\beta} lm \cos C = 0.$$

This then is the tangential equation required.

It will be seen that

$$\Omega = \left(\frac{a}{\alpha} l - \frac{b}{\beta} e^{iC} m - \frac{c}{\gamma} e^{-iB} n\right) \left(\frac{a}{\alpha} l - \frac{b}{\beta} e^{-iC} m - \frac{c}{\gamma} e^{iB} n\right).$$

So that the coordinates of the circular points are proportional to

$$\left\{\frac{a}{\alpha}, \quad -\frac{b}{\beta}(\cos C + i \sin C), \quad -\frac{c}{\gamma}(\cos B - i \sin B)\right\},$$

and $\left\{\frac{a}{\alpha}, \quad -\frac{b}{\beta}(\cos C - i \sin C), \quad -\frac{c}{\gamma}(\cos B + i \sin B)\right\}.$

These coordinates are not however of any serious importance. It is the expression Ω that will be used hereafter.

384. Circular points in areal and trilinear coordinates.

(i) If the point coordinates be areal, $\alpha = \beta = \gamma = 1$, and we get

$$\Omega \equiv a^2l^2 + b^2m^2 + c^2n^2 - 2bcmn \cos A$$
$$- 2canl \cos B - 2ablm \cos C = 0.$$

(ii) If the point coordinates be trilinear, $\alpha : \beta : \gamma = a : b : c$, and the equation simplifies down to

$$\Omega \equiv l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C = 0.$$

It is thus a special advantage which the system of trilinear coordinates has, that the equation of the circular points reduces to so simple a form.

Example. Shew that the length of the perpendicular distance of the point (x, y, z) from the line $L \equiv lx + my + nz = 0$ in generalised homogeneous coordinates is $\frac{2\Delta L}{\sqrt{\Omega}}$ where Δ is the area of the triangle of reference.

385. Circular points in Cartesian coordinates.

To find the tangential equation of the circular points at infinity in Cartesian coordinates.

The equation of the circle of radius a having its centre at the origin is

$$x^2 + 2xy\cos\omega + y^2 - a^2z^2 = 0 \quad\ldots\ldots\ldots\ldots(1).$$

The line at infinity is $\quad z = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$

We require the condition that the line

$$lx + my + nz = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

should pass through a point of intersection of (1) and (2).

Using (2) in (1) and (3) we have

$$x^2 + 2xy\cos\omega + y^2 = 0,$$

$$lx + my = 0,$$

$$\therefore\ l^2 - 2lm\cos\omega + m^2 = 0 \quad\ldots\ldots\ldots\ldots(4).$$

This then is the equation required. It reduces to

$$l^2 + m^2 = 0,$$

when the axes are rectangular.

Expressing (4) in factors we have

$$(l - me^{\omega i})(l - me^{-\omega i}) = 0.$$

That is the two circular points at infinity have for their separate equations

$$l - me^{\omega i} + 0 \,.\, n = 0, \qquad l - me^{-\omega i} + 0 \,.\, n = 0.$$

Their coordinates are therefore $(t, -e^{\omega i}t)$ and $(t, -e^{-\omega i}t)$ where t is infinitely large. When the axes are rectangular these are $(t, -it)$, $(t, +it)$.

386. Confocal conics.

We will make use of the tangential equation of the circular points at infinity *to find the equation of the system of conics confocal with*

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad\ldots\ldots(1),$$

supposed referred to rectangular Cartesian axes (so that $z = 1$).

The lines joining the foci of a conic to the circular points touch the conic (*Pure Geometry*, § 250).

Therefore the conics confocal with (1) touch the tangents drawn from the circular points to (1).

Thus if

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

be the tangential equation of the conic, since the tangential equation of the circular points is

$$l^2 + m^2 = 0,$$

the tangential equation of the conics confocal with the given conic is

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm + \lambda(l^2 + m^2) = 0,$$

i.e. $(A + \lambda) l^2 + (B + \lambda) m^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$

Thus their Cartesian equation is

$$\begin{vmatrix} A+\lambda, & H, & G, & x \\ H, & B+\lambda, & F, & y \\ G, & F, & C, & z \\ x, & y, & z, & 0 \end{vmatrix} = 0,$$

that is

$$\begin{vmatrix} A, & H, & G, & x \\ H, & B, & F, & y \\ G, & F, & C, & z \\ x, & y, & z, & 0 \end{vmatrix} + \begin{vmatrix} \lambda, & H, & G, & x \\ 0, & B, & F, & y \\ 0, & F, & C, & z \\ 0, & y, & z, & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} A, & 0, & G, & x \\ H, & \lambda, & F, & y \\ G, & 0, & C, & z \\ x, & 0, & z, & 0 \end{vmatrix} + \begin{vmatrix} \lambda, & 0, & G, & x \\ 0, & \lambda, & F, & y \\ 0, & 0, & C, & z \\ 0, & 0, & z, & 0 \end{vmatrix} = 0,$$

that is

$$-\Delta S + \lambda \begin{vmatrix} B, & F, & y \\ F, & C, & z \\ y, & z, & 0 \end{vmatrix} + \lambda \begin{vmatrix} A, & G, & x \\ G, & C, & z \\ x, & z, & 0 \end{vmatrix} + \lambda^2 \begin{vmatrix} C, & z \\ z, & 0 \end{vmatrix} = 0,$$

that is

$$-\Delta S+\lambda(2Fyz-Cy^2-Bz^2+2Gzx-Cx^2-Az^2)-\lambda^2z^2=0,$$

that is

$$\Delta S+\lambda\{C(x^2+y^2)-2Fyz-2Gzx+(A+B)z^2\}+\lambda^2z^2=0.$$

387. The foci.

To find the foci of the conic

$$S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

referred to rectangular Cartesian axes.

Form the tangential equation

$$\Sigma\equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

Then $$\Sigma+k(l^2+m^2)=0$$

represents the conics touching the tangents from the circular points to the given conic, which tangents intersect in the foci.

Determine k so that this is the product of two linear factors in l, m, n and the two points thus determined will be the foci.

The equation giving k is

$$\begin{vmatrix} A+k, & H, & G \\ H, & B+k, & F \\ G, & F, & C \end{vmatrix}=0,$$

that is

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}+\begin{vmatrix} k, & H, & G \\ 0, & B, & F \\ 0, & F, & C \end{vmatrix}$$

$$+\begin{vmatrix} A, & 0, & G \\ H, & k & F, \\ G, & 0, & C \end{vmatrix}+\begin{vmatrix} k, & 0, & G \\ 0, & k, & F \\ 0, & 0, & C \end{vmatrix}=0,$$

that is $$\Delta^2+k(BC-F^2+AC-G^2)+k^2C=0,$$

that is $$\Delta^2+k\Delta(a+b)+k^2(ab-h^2)=0.$$

In the case where the conic is a parabola this reduces to

$$k(a+b)+\Delta=0.$$

Examples. 1. Find all the foci of the conic whose equation referred to rectangular axes is

$$2x^2 - 8xy - 4y^2 - 4y + 1.$$

[Here $a=2,\ b=-4,\ c=1,\ f=-2,\ g=0,\ h=-4$.

$\therefore\quad A=-8,\ B=2,\ C=-24,\ F=4,\ G=8,\ H=4$ and $\Delta=-32$.

The equation giving k is

$$-24k^2 - 2k\Delta + \Delta^2 = 0,$$

that is
$$(6k-\Delta)(4k+\Delta)=0,$$

which gives
$$k=\frac{\Delta}{6}=-\frac{16}{3}, \text{ or } k=-\frac{\Delta}{4}=8.$$

Taking $k=8$, we have

$$\begin{aligned}\Sigma + k\Omega &= -8l^2+2m^2-24n^2+8mn+16nl+8lm+8(l^2+m^2)\\ &=10m^2-24n^2+8mn+16nl+8lm\\ &=2(m+2n)(4l+5m-6n).\end{aligned}$$

Thus we have the two foci whose coordinates are proportional to

$$(0,\ 1,\ 2),\quad (4,\ 5,\ -6).$$

Hence as $z=1$, the coordinates of the foci are $(0, \frac{1}{2})$, $(-\frac{2}{3}, -\frac{5}{6})$.

Next taking $k=-\frac{16}{3}$, we have

$$\begin{aligned}3(\Sigma+k\Omega) &= -24l^2+6m^2-72n^2+24mn+48nl+24lm-16(l^2+m^2)\\ &=-40l^2-10m^2-72n^2+24mn+48nl+24lm\\ &=-2(20l^2+5m^2+36n^2-12mn-24nl-12lm).\end{aligned}$$

The imaginary foci are then given by

$$20l^2+5m^2+36n^2-12mn-24nl-12lm=0,$$

that is
$$20l^2-2(6m+12n)l+5m^2-12mn+36n^2=0.$$

$$\therefore\quad l=\frac{6m+12n\pm\sqrt{(6m+12n)^2-20(5m^2-12mn+36n^2)}}{20}$$

$$=\frac{6m+12n\pm\sqrt{-64m^2+384mn-576n^2}}{20}=\frac{6m+12n\pm i(8m-24n)}{20}.$$

$$\therefore\quad 10l=(3\pm 4i)m+(6\mp 12i)n$$

that is

$$10l-(3+4i)m-(6-12i)n=0, \text{ or } 10l-(3-4i)m-(6+12i)n=0;$$

thus the coordinates of the foci are

$$\left\{-\frac{10}{6(1-2i)},\ \frac{3+4i}{6(1-2i)}\right\} \text{ and } \left\{\frac{-10}{6(1+2i)},\ \frac{3-4i}{6(1+2i)}\right\},$$

that is
$$\left\{-\frac{1+2i}{3},\ \frac{-1+2i}{6}\right\} \text{ and } \left\{-\frac{1-2i}{3},\ -\frac{1+2i}{6}\right\}.$$

It can be verified that the line joining these two foci and that joining the two real ones bisect one another at right angles.]

2. Find the focus of the parabola

$$x^2+2xy+y^2-4x+8y-6=0.$$

3. Find all the foci of the conic

$$8x^2-4xy+5y^2-16x-14y+17=0.$$

388. Condition for a rectangular hyperbola.

To find by means of the circular points at infinity the condition that the general equation of the second degree in generalised homogeneous coordinates may be a rectangular hyperbola.

We will take as the general equation

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0\ldots\ldots(1).$$

The pair of points (x_1, y_1, z_1) (x_2, y_2, z_2) will be conjugate for this if

$$Ax_1x_2+By_1y_2+Cz_1z_2+F(y_1z_2+y_2z_1)+G(z_1x_2+z_2x_1)$$
$$+H(x_1y_2+x_2y_1)=0,$$

and thus the pair of points given by

$$ul^2+vm^2+wn^2+2u'mn+2v'nl+2w'lm=0$$

in tangential coordinates will be conjugate for (1) if

$$Au+Bv+Cw+2Fu'+2Gv'+2Hw'=0.$$

Now the circular points at infinity are given by

$$\frac{a^2}{\alpha^2}l^2+\frac{b^2}{\beta^2}m^2+\frac{c^2}{\gamma^2}n^2-2\frac{bc}{\beta\gamma}mn\cos A-2\frac{ca}{\gamma\alpha}nl\cos B$$
$$-2\frac{ab}{\alpha\beta}lm\cos C=0.$$

Therefore the circular points will be conjugate for (1), that is the conic will be a rectangular hyperbola if (see *Pure Geometry*, § 248)

$$A\frac{a^2}{\alpha^2}+B\frac{b^2}{\beta^2}+C\frac{c^2}{\gamma^2}-2F\frac{bc}{\beta\gamma}\cos A-2G\frac{ca}{\gamma\alpha}\cos B$$
$$-2H\frac{ab}{\alpha\beta}\cos C=0.$$

This agrees with our previously obtained condition (§ 311).

389. Perpendicular lines.

To find the condition that two lines

$$lx + my + nz = 0, \quad l'x + m'y + n'z = 0,$$

should be perpendicular.

We require the condition that

$$(lx + my + nz)(l'x + m'y + n'z) = 0$$

should satisfy the condition for a rectangular hyperbola obtained in the last paragraph.

This condition gives

$$ll'\frac{a^2}{\alpha^2} + mm'\frac{b^2}{\beta^2} + nn'\frac{c^2}{\gamma^2} - (mn' + m'n)\frac{bc}{\beta\gamma}\cos A$$
$$- (nl' + n'l)\frac{ca}{\gamma\alpha}\cos B - (lm' + l'm)\frac{ab}{\alpha\beta}\cos C = 0.$$

And this can be written

$$l'\frac{\partial\Omega}{\partial l} + m'\frac{\partial\Omega}{\partial m} + n'\frac{\partial\Omega}{\partial n} = 0,$$

or its equivalent

$$l\frac{\partial\Omega'}{\partial l'} + m\frac{\partial\Omega'}{\partial m'} + n\frac{\partial\Omega'}{\partial n'} = 0.$$

EXAMPLES.

1. Obtain the coordinates of the centre of the conic whose tangential equation for Cartesian coordinates is $(a, b, c, f, g, h)(l, m, n)^2$ and shew that the conic is a parabola if $c = 0$.

2. The line $lx + my + n = 0$ satisfies the relation

$$(a, b, c, f, g, h)(l, m, n)^2 = 0,$$

shew that its envelope is an ellipse, parabola, or hyperbola according as

$$c\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \gtreqless 0.$$

3. The envelope of chords of an ellipse which subtend a right angle at the centre is a concentric circle.

4. Obtain in generalised homogeneous coordinates the tangential equation of a circle whose centre is (f, g, h) and whose radius is r in the form

$$4\Delta^2 (lf + mg + nh)^2 = r^2 \left\{\frac{a^2}{\alpha^2} l^2 + \frac{b^2}{\beta^2} m^2 + \frac{c^2}{\gamma^2} n^2 - 2\frac{bc}{\beta\gamma} mn \cos A - 2\frac{ca}{\gamma\alpha} nl \cos B - 2\frac{ab}{\alpha\beta} lm \cos C\right\},$$

Δ being the area of the triangle of reference.

Prove that the conics

$$yz + zx + xy = 0$$

and
$$\sin\frac{A}{2}\sqrt{x} + \sin\frac{B}{2}\sqrt{y} + \sin\frac{C}{2}\sqrt{z} = 0$$

in *trilinears* are confocal.

[Use the method of § 386. Form the tangential equations $\Sigma = 0$, $\Sigma' = 0$ and shew that $\Sigma' = \Sigma + k\Omega$.]

6. The condition that the line $lx + my + n = 0$ should be a normal to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the axes being rectangular, is

$$(al^2 + 2hlm + bm^2)(Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm) = \Delta (l^2 + m^2)^2.$$

[Write down the equation of the pair of tangents at the points where the line cuts the conic (§ 368). Then get equation of pair of lines through the origin parallel to these, and express the fact that $lx + my = 0$ is perpendicular to one of these.]

7. If $(a, b, c, f, g, h)(l, m, n)^2 = 0$ be the tangential equation of a conic, the coordinates of the pole of the line $lx + my + nz = 0$ with respect to it are in the ratio

$$al + hm + gn : hl + bm + fn : gl + fm + cn.$$

8. The conics confocal with $ax^2 + 2hxy + by^2 = 1$ are given by

$$(a + \lambda) x^2 + 2hxy + (b + \lambda) y^2 = 1 + \lambda (\lambda + a + b) \div (ab - h^2),$$

the axes being rectangular.

9. Tangents are drawn at the feet of the normals from a point (f, g) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Shew that the equation of the parabola which touches the four tangents is

$$\sqrt{fx}+\sqrt{-gy}+\sqrt{a^2-b^2}=0.$$

[The tangent at (x_1, y_1) is $lx+my+n=0$ where

$$l:m:n=\frac{x_1}{a^2}:\frac{y_1}{b^2}:-1.$$

The normal at this point goes through (f, g) if

$$\frac{a^2f}{x_1}-\frac{b^2g}{y_1}=a^2-b^2,$$

$$\therefore\ \frac{f}{l}-\frac{g}{m}+\frac{a^2-b^2}{n}=0.$$

Hence the tangent touches the conic whose point equation is

$$\sqrt{fx}+\sqrt{-gy}+\sqrt{(a^2-b^2)z}=0,$$

but this conic is inscribed in the triangle of reference and therefore touches $z=0$ which in this case is the line at infinity. Thus the four tangents touch the parabola

$$\sqrt{fx}+\sqrt{-gy}+\sqrt{a^2-b^2}=0.$$

This parabola moreover touches the coordinate axes.]

10. If two conics have three point contact, and Q be the pole with respect to the second of the tangent at P on the first, the envelope of PQ is a conic having double contact with the first.

[Take as triangle of reference that formed by the common tangents and common chord.]

11. From (x_0, y_0) four normals are drawn to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$; shew that the four corresponding tangents satisfy the tangential equation

$$x_0\mu\nu-y_0\nu\lambda+c^2\lambda\mu=0 \text{ where } c^2\equiv a^2-b^2.$$

Thence by considering the point equation of

$$k(a^2\lambda^2+b^2\mu^2-\nu^2)+2[x_0\mu\nu-y_0\nu\lambda+c^2\lambda\mu]=0$$

shew that the equation of the four tangents is

$$(c^2xy-a^2x_0y+b^2y_0x)^2-(b^2x^2+a^2y^2-a^2b^2)[(xx_0-yy_0-c^2)^2+4xyx_0y_0]=0.$$

12. Shew that the envelope of polars of a point with respect to a series of confocal conics is a parabola which touches the axes; and prove that if normals be drawn from the point to the conics, the tangents at their feet touch this parabola.

13. Shew that the asymptotes of the hyperbolas which touch the axes where $lx + my - 1 = 0$ cuts them envelope the parabola

$$(lx - my)^2 - 4(lx + my - 1) = 0.$$

14. From a point T (α, β) tangents TP, TQ are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Shew that the equation of the parabola which has double contact with the ellipse at P and Q is

$$(x\beta - y\alpha)^2 + 2b^2\alpha x + 2a^2\beta y = b^2\alpha^2 + a^2\beta^2 + a^2b^2.$$

Shew also that if (α, β) is on the line $lx + my = 1$ the directrix of the parabola envelopes the parabola whose equation is

$$(mx - ly)^2 + 2(lx + my) = 1 + (l^2 + m^2)(a^2 + b^2).$$

15. The sides of a triangle inscribed in an ellipse touch a confocal ellipse. Shew that the points of contact are the points at which they touch the corresponding ecircles.

16. A straight line meets one of a system of confocal conics in P and Q, and RS is the line joining the feet of the other two normals drawn from the point of intersection of the normals at P and Q. Prove that the envelope of RS is a parabola touching the axes.

17. Two fixed points P and Q are taken on a given conic, and R is any point on a fixed straight line; the lines PR and QR meet the conic again in P' and Q'; prove that the envelope of $P'Q'$ is a conic.

18. If two conics have double contact, then every conic confocal with the first has double contact with some one confocal to the second, and the four common tangents to any confocal to the first and any confocal to the second touch a variable conic which touches the common tangents of the original conics.

19. The envelope of a chord of a conic subtending a constant angle at a focus of the conic is another conic with the same focus and axis.

20. A straight line passes through a given point. Prove that the envelope of the line joining its poles with respect to two given coaxial conics is a parabola.

21. From points on a given straight line lines are drawn parallel to the polars of the points with respect to a conic; prove that these lines envelope a parabola.

22. A system of conics have a common focus and directrix. Shew that the normals at the points where a line through the focus in a given direction meets the conics envelope a parabola having its vertex at the focus and touching the given line.

23. Interpret the equations in generalised homogeneous coordinates:

(i) $A\,(lx+my+nz)^2 + B\,(l'x+m'y+n'z)^2 = C(\alpha x+\beta y+\gamma z)^2$,

(ii) $(lx+my+nz)^2 = k\,(l'x+m'y+n'z)\,(\alpha x+\beta y+\gamma z)$,

(iii) $(lx+my+nz)\,(l'x+m'y+n'z) = k\,(\alpha x+\beta y+\gamma z)^2$,

in the cases (*a*) where l, m, n, l', m', n' are connected by the relation

$$l'\frac{\partial\Omega}{\partial l} + m'\frac{\partial\Omega}{\partial m} + n'\frac{\partial\Omega}{\partial n} = 0,$$

(*b*) where they are not so connected.

24. A parabola circumscribes a triangle, shew that its axis touches a curve of the third class given by

$$\lambda^2\frac{\partial\Omega}{\partial\lambda} + \mu^2\frac{\partial\Omega}{\partial\mu} + \nu^2\frac{\partial\Omega}{\partial\nu} = 0,$$

and the tangent at the vertex always touches the curve of the sixth class defined by the rationalised form of

$$\lambda^{\frac{1}{2}}\frac{\partial\Omega}{\partial\lambda} + \mu^{\frac{1}{2}}\frac{\partial\Omega}{\partial\mu} + \nu^{\frac{1}{2}}\frac{\partial\Omega}{\partial\nu} = 0,$$

where the line at infinity in point coordinates is $x+y+z=0$, and the rational tangential equation of the circular points at infinity is $\Omega = 0$.

[Use Ex. 23.]

25. Consider what modification is needed in Ex. 24 if the equation of the line at infinity be $\alpha x+\beta y+\gamma z=0$.

26. The point coordinates being areal, prove that the asymptotes of all conics circumscribing the fundamental triangle and passing through the point (x_1, y_1, z_1) touch a curve of the third class whose tangential equation is

$$\frac{(\mu-\nu)^2\lambda}{x_1}+\frac{(\nu-\lambda)^2\mu}{y_1}+\frac{(\lambda-\mu)^2\nu}{z_1}=0.$$

27. If four normals be drawn to the conic whose tangential equation is $\Sigma = 0$ from (x_1, y_1, z_1), then the tangential equation of the parabola touching the tangents to the conic at the four feet is

$$x_1\frac{\partial(\Sigma, \Omega)}{\partial(\mu, \nu)}+y_1\frac{\partial(\Sigma, \Omega)}{\partial(\nu, \lambda)}+z_1\frac{\partial(\Sigma, \Omega)}{\partial(\lambda, \mu)}=0,$$

where $\Omega = 0$ is the rational tangential equation of the circular points at infinity.

28. A conic confocal with the conic S exists which touches the sides of the triangle ABC. Shew that of the conics which touch the four tangents to S from the points B and C, one has a focus at A and the companion focus on BC.

29. Find the envelope of a line on which two circles intercept chords whose lengths bear a constant ratio to each other.

30. A chord PQ of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is drawn through the fixed point (f, g). Prove that if the circle through P, Q and C, the centre of the ellipse, cut the curve again in the points R and S, then will RS touch the parabola whose focus is C, and the equation of the tangent at the vertex

$$(a^2-b^2)(gy-fx)+a^2b^2=0.$$

31. If a parabola passes through fixed points A, B, C, the envelope of the tangent to it at the extremity of the diameter through a fixed point D is a conic circumscribing the triangle ABC. [Use Ex. 23.]

32. If $\Omega = 0$ be the tangential equation of the circular points at infinity, then the line $\lambda x+\mu y+\nu z=0$ is a normal to the conic

$$(a, b, c, f, g, h)(x, y, z)^2=0,$$

provided that

$$(A, B, C, F, G, H)\left\{\frac{\partial(\Sigma, \Omega)}{\partial(\mu, \nu)}, \frac{\partial(\Sigma, \Omega)}{\partial(\nu, \lambda)}, \frac{\partial(\Sigma, \Omega)}{\partial(\lambda, \mu)}\right\}^2=0.$$

33. The point coordinates being areal, shew that the equation of the director circle of the conic (inscribed in the triangle of reference) whose tangential equation is

$$fmn + gnl + hlm = 0$$

is $$\{(b^2 + c^2 - a^2) fx + (c^2 + a^2 - b^2) gy + (a^2 + b^2 - c^2) hz\} (x + y + z)$$
$$= 2 (f + g + h) (a^2yz + b^2zx + c^2xy).$$

34. Find the equation of the ellipse whose real foci referred to rectangular axes are (x_1, y_1), (x_2, y_2) and whose minor axis has length $2k$, in the form

$$[(x - x_1)(y - y_2) - (x - x_2)(y - y_1)]^2$$
$$+ 4k^2 [(x - x_1)(x - x_2) + (y - y_1)(y - y_2)] - 4k^4 = 0$$

CHAPTER XIX.

COVARIANTS.

390. Covariants defined.

In the last chapter (§ 381) we saw that if

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \text{ ......(1)},$$

$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0 \text{...(2)},$$

be the equations of two conics in any system of homogeneous coordinates, then

$$\mathbf{F}^2 - 4\Delta\Delta' SS' = 0$$

is the equation of their four common tangents.

Now suppose that we transform our equations to any other system of homogeneous coordinates by the substitutions

$$\left.\begin{aligned} x &= \lambda_1 X + \mu_1 Y + \nu_1 Z \\ y &= \lambda_2 X + \mu_2 Y + \nu_2 Z \\ z &= \lambda_3 X + \mu_3 Y + \nu_3 Z \end{aligned}\right\} \text{..................(A)}.$$

So that

$$S = S_1 \equiv (a_1, b_1, c_1)(X, Y, Z)^2 = 0 \text{ ....(3)},$$

$$S' = S_1' \equiv (a_1', b_1', c_1')(X, Y, Z)^2 = 0 \text{ ....(4)}.$$

Then the equation of the four common tangents to $S_1 = 0$, $S_1' = 0$ will be

$$\mathbf{F}_1^2 - 4\Delta_1\Delta_1' S_1 S_1' = 0,$$

where Δ_1, Δ_1', $\mathbf{F}_1$, S_1, S_1' are exactly the same functions of the new coefficients and of the new variables that Δ, Δ', $\mathbf{F}$, S, S' were of the old coefficients and variables.

But we could find the equation of the four common tangents to (3) and (4), by taking the equation

$$\mathbf{F}^2 - 4\Delta\Delta' SS' = 0,$$

which represents the tangents to the same two conics in the forms (1) and (2), and then changing the variables by the relations (A), afterwards expressing the coefficients $a, b, \ldots a', b'$, etc. in terms of the new coefficients.

The result must be the same in the two cases.

This however does not entitle us to say that

$$\mathbf{F}_1^2 - 4\Delta_1\Delta_1'S_1S_1' = \mathbf{F}^2 - 4\Delta\Delta'SS'.$$

What we can infer is that the ratio

$$\mathbf{F}_1^2 - 4\Delta_1\Delta_1'S_1S_1' : \mathbf{F}^2 - 4\Delta\Delta'SS'$$

is independent of the variables. And as $S_1 = S$ and $S_1' = S'$ while $\Delta_1 = \epsilon^2\Delta$ and $\Delta_1' = \epsilon^2\Delta'$, where

$$\epsilon \equiv \begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \lambda_3, & \mu_3, & \nu_3 \end{vmatrix},$$

(for this has been proved in § 355) it is clear that $\mathbf{F}_1 = \epsilon^2\mathbf{F}$.

This could indeed be proved by actual substitution but the work would be very laborious.

The expression $\mathbf{F}$ and likewise the expression

$$\mathbf{F}^2 - 4\Delta\Delta'SS'$$

are called *covariants* for the two conics.

Definition. A covariant then may be defined as a function of the variables and coefficients in the equations of two conics, such that the same function of the new variables and the new coefficients of the same two conics, when their equations are transformed by linear substitutions, bears to the original function a constant ratio. This constant ratio is always as a matter of fact a power of the determinant ϵ.

391. Geometrical interpretation of covariants.

It seems clear then that any covariant of two conics equated to zero gives a locus connected geometrically with the two conics. For example, as we have seen, $\mathbf{F}^2 - 4\Delta\Delta'SS' = 0$ is the equation of the four common tangents. We naturally ask then how the locus $\mathbf{F} = 0$, which is a conic, is connected with the

two conics. Salmon has proved that this is the locus of points, the two pairs of tangents from which to the two conics are harmonically conjugate to each other.

This we shall now proceed to prove, nor shall we in so doing assume the covariant character of **F**.

392. The F conic.

To find the locus of points the pairs of tangents from which to two given conics

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \;\ldots\ldots(1),$$

$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0 \quad (2),$$

are harmonically conjugate to each other.

The equation of the pair of tangents from (x_1, y_1, z_1) to (1) is

$$\begin{aligned}(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)&\\ \times (ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1)&\\ - \{(ax_1 + hy_1 + gz_1)\,x + (hx_1 + by_1 + fz_1)\,y&\\ + (gx_1 + fy_1 + cz_1)\,z\}^2 &= 0.\end{aligned}$$

Now the coefficient of x^2 on the left side is

$$a\,(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) - (ax_1 + hy_1 + gz_1)^2,$$

which $= (ab - h^2)\,y_1^2 + (ca - g^2)\,z_1^2 - 2\,(gh - af)\,y_1z_1$

$= Cy_1^2 + Bz_1^2 - 2Fy_1z_1.$

Similarly the coefficient of y^2 is

$$Az_1^2 + Cx_1^2 - 2Gz_1x_1,$$

and the coefficient of z^2 is

$$Bx_1^2 + Ay_1^2 - 2Hx_1y_1.$$

Also the coefficient of $2yz$ is

$$f\,(ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) - (hx_1 + by_1 + fz_1)\,(gx_1 + fy_1 + cz_1),$$

which

$$= -(gh - af)\,x_1^2 - (bc - f^2)\,y_1z_1 + (fg - ch)\,z_1x_1 + (hf - bg)\,x_1y_1$$

$$= -Fx_1^2 - Ay_1z_1 + Hz_1x_1 + Gx_1y_1.$$

Similarly the coefficient of $2zx$ will be

$$-Gy_1^2 - Bz_1x_1 + Fx_1y_1 + Hy_1z_1,$$

and the coefficient of $2xy$ will be

$$-Hz_1^2 - Cx_1y_1 + Gy_1z_1 + Fz_1x_1.$$

Thus the equation of the pair of tangents is

$$(Cy_1^2 + Bz_1^2 - 2Fy_1z_1)\,x^2 + \text{two similar terms}$$
$$+ 2\,(Gx_1y_1 + Hz_1x_1 - Ay_1z_1 - Fx_1^2)\,yz + \text{two similar terms} = 0.$$

Putting $z = 0$ in this we obtain

$$(Cy_1^2 + Bz_1^2 - 2Fy_1z_1)\,x^2 + (Az_1^2 + Cx_1^2 - 2Gz_1x_1)\,y^2$$
$$+ 2\,(Fz_1x_1 + Gy_1z_1 - Cx_1y_1 - Hz_1^2)\,xy = 0 \quad \text{............(3)},$$

which is the equation of a pair of lines joining the C-vertex of the triangle of reference to the points where the pair of tangents meet the opposite side of the triangle of reference.

Similarly

$$(C'y_1^2 + B'z_1^2 - 2F'y_1z_1)\,x^2 + (A'z_1^2 + C'x_1^2 - 2G'z_1x_1)\,y^2$$
$$+ 2\,(F'z_1x_1 + G'y_1z_1 - C'x_1y_1 - H'z_1^2)\,xy = 0 \quad \text{.........(4)}$$

will be the equation of the pair of lines joining C to the points where the pair of tangents to (2) meet the opposite sides of the triangle of reference.

The condition that the two pairs of tangents should be harmonically conjugate is plainly the same as that (3) and (4) should be so, and this is

$$(Cy_1^2 + Bz_1^2 + 2Fy_1z_1)(A'z_1^2 + C'x_1^2 - 2G'z_1x_1)$$
$$+ (C'y_1^2 + B'z_1^2 - 2F'y_1z_1)(Az_1^2 + Cx_1^2 - 2Gz_1x_1)$$
$$= 2\,(Fz_1x_1 + Gy_1z_1 - Cx_1y_1 - Hz_1^2)$$
$$\times (F'z_1x_1 + G'y_1z_1 - C'x_1y_1 - H'z_1^2).$$

Multiplying this out and dividing by z_1^2 we see that the locus of (x_1, y_1, z_1) is the conic

$$(BC' + B'C - 2FF')\,x^2 + \text{two similar terms}$$
$$+ 2\,(GH' + G'H - AF' - A'F)\,yz + \text{two similar terms} = 0,$$

that is $$\mathbf{F} = 0.$$

393. The work of the preceding paragraph could be made easier if we were to assume the covariant character of F, for then we could reduce the equations of the two conics to standard forms and write

$$S \equiv a_1X^2 + b_1Y^2 + c_1Z^2,$$

$$S' \equiv a_1'X^2 + b_1'Y^2 + c_1'Z^2.$$

The algebra would then be simpler, and we should find that the locus of the points, the pairs of tangents from which to the two conics are mutually harmonically conjugate, is

$$\mathbf{F}_1 = 0.$$

And as $\mathbf{F}_1 = \epsilon^2\mathbf{F}$, we see that the locus when we revert to the old coordinates is $\mathbf{F} = 0$.

The student can gain practice in the algebra by going through the work in this simpler case.

394. Properties of the F conic.

The conic $\mathbf{F} = 0$ *passes through the eight points of contact of the common tangents to* $S = 0$, $S' = 0$.

For let P be a point of contact with S of a common tangent to the two conics. If the equation of this common tangent be $lx + my + nz = 0$, then the pair of tangents from P to S will be

$$(lx + my + nz)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the pair of tangents from P to S' will be (say)

$$(lx + my + nz)(l'x + m'y + n'z) = 0 \quad \ldots\ldots\ldots(2),$$

and the lines (1) satisfy the test for being harmonically conjugate with (2).

Hence P lies on the locus $\mathbf{F} = 0$.

Cor. 1. If $S = 0$, $S' = 0$ touch, then $\mathbf{F} = 0$ will touch them both at their point, or points, of contact.

Cor. 2. If S and S' have three point contact at P, $\mathbf{F}$ will also have three point contact with them at P.

Cor. 3. If S and S' have four point contact at P, $\mathbf{F}$ will also have four point contact with them at P.

Examples. 1. $S \equiv cz^2 + 2hxy = 0$ and $S' \equiv by^2 + cz^2 + 2hxy = 0$ are two conics with four point contact. Prove that their **F** conic is

$$by^2 + 2cz^2 + 4hxy = 0,$$

and so verify that **F** has four point contact with S and S' at their common point $y = 0$, $z = 0$.

2. $S \equiv cz^2 + 2hxy = 0$ and $S' \equiv cz^2 + 2fyz + 2hxy = 0$ are two conics with three point contact at $y = 0$, $z = 0$. Express their **F** conic in the form

$$-\tfrac{1}{2} f^2 y^2 + c\,(cz^2 + 2fyz + 2hxy) = 0,$$

and shew that it has three point contact with them.

395. F in the standard form.

The covariant **F** assumes a simple form when the equations of the two conics are given in the form

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv x^2 + y^2 + z^2 = 0,$$

for then $\quad A = bc, \quad B = ca, \quad C = ab, \quad F = G = H = 0,$

$$A' = 1, \quad B' = 1, \quad C' = 1, \quad F' = G' = H' = 0.$$

So that $F \equiv a\,(b + c)\,x^2 + b\,(c + a)\,y^2 + c\,(a + b)\,z^2$.

We see then that the conic $\mathbf{F} = 0$ has for a self-conjugate triangle any triangle self-conjugate to the original conics.

396. Conditions for double contact.

To find the conditions that the conics

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0,$$

should have double contact.

We have seen (§ 324) that if two conics have double contact their equations can be reduced to the forms

$$S = \overline{S} \equiv a\,(x^2 + y^2) + cz^2 = 0,$$
$$S' = \overline{S}' \equiv x^2 + y^2 + z^2 = 0,$$

whence

$$\overline{\mathbf{F}} \equiv a\,(a + c)\,x^2 + a\,(c + a)\,y^2 + c\,(2a)\,z^2 = a\overline{S} + ac\overline{S}'.$$

Thus $\overline{\mathbf{F}}$ is a linear function of $\overline{S}$ and $\overline{S}'$.

Therefore since $\bar{S} = S$ and $\bar{S}' = S'$, and $\bar{\mathbf{F}} = \epsilon^2 \mathbf{F}$, we see that $\mathbf{F}$ must be a linear function of S and S'; so then we write

$$\mathbf{F} = kS + lS'.$$

Now let us write $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h} \between x, y, z)^2$ for $\mathbf{F}$, so that

$$\mathrm{a} \equiv BC' + B'C - 2FF',$$

and so on.

We thus get

$$\mathrm{a} = ka + la', \quad \mathrm{b} = kb + lb', \quad \mathrm{c} = kc + lc',$$
$$\mathrm{f} = kf + lf', \quad \mathrm{g} = kg + lg', \quad \mathrm{h} = kh + lh'.$$

Hence we have as the conditions for double contact of the conics S and S',

$$\left\| \begin{matrix} \mathrm{a}, & \mathrm{b}, & \mathrm{c}, & \mathrm{f}, & \mathrm{g}, & \mathrm{h} \\ a, & b, & c, & f, & g, & h \\ a', & b', & c', & f', & g', & h' \end{matrix} \right\| = 0,$$

by which is meant that all the determinants formed by taking any three of these columns are zero.

The student will see also that these conditions are *sufficient* to ensure double contact.

397. Polar reciprocal.

We may use the covariant $\mathbf{F}$ and the invariants to find the equation of a conic geometrically related to two given conics.

Thus let us find the equation of the polar reciprocal of

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

with respect to

$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0.$$

Transform the conics to

$$S = \bar{S} \equiv \alpha X^2 + \beta Y^2 + \gamma Z^2 = 0,$$
$$S' = \bar{S}' \equiv X^2 + Y^2 + Z^2 = 0.$$

We then have

$$\bar{\Delta} = \alpha\beta\gamma, \quad \bar{\Delta}' = 1, \quad \bar{\Theta} = \beta\gamma + \gamma\alpha + \alpha\beta, \quad \bar{\Theta}' = \alpha + \beta + \gamma$$

and $\qquad \bar{\mathbf{F}} = \alpha(\beta + \gamma) X^2 + \beta(\gamma + \alpha) Y^2 + \gamma(\alpha + \beta) Z^2.$

Then (§ 320) the polar reciprocal of $\bar{S}$ with respect to $\bar{S}'$ is

$$\beta\gamma X^2 + \gamma\alpha Y^2 + \alpha\beta Z^2 = 0,$$

that is

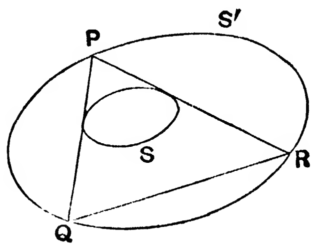
$$(\beta\gamma + \gamma\alpha + \alpha\beta)(X^2 + Y^2 + Z^2)$$
$$- \{\alpha(\beta+\gamma) X^2 + \beta(\gamma+\alpha) Y^2 + \gamma(\alpha+\beta) Z^2\} = 0,$$

that is $$\bar{\Theta}\bar{S}' - \bar{\mathrm{F}} = 0.$$

Hence the polar reciprocal of S with respect to S' is

$$\Theta S' - \mathrm{F} = 0.$$

Examples. 1. *S and S' are two conics; from P a point on S' tangents are drawn to S to meet S' again in Q and R; it is required to find the envelope of the line QR.*



The first solution we will give is somewhat lengthy but it is instructive from an analytical point of view.

We will transform the equations of our conics so that

$$S = x^2 + y^2 + z^2 \quad\text{..............................(1)},$$
$$S' = ax^2 + by^2 + cz^2 \quad\text{........................(2)}.$$

We may represent a point on (2) by means of the relations

$$\sqrt{a}x = z\sqrt{-c}\cos\theta,$$
$$\sqrt{b}y = z\sqrt{-c}\sin\theta,$$

and then the equation of the chord through the points given by $\theta = \alpha$ and $\theta = \beta$, will be

$$\sqrt{a}x\cos\frac{\alpha+\beta}{2} + \sqrt{b}y\sin\frac{\alpha+\beta}{2} - z\sqrt{-c}\cos\frac{\alpha-\beta}{2} = 0 \quad\text{......(3)}.$$

Let Q and R be the points α and β, and let P be given by $\theta = \gamma$. Then the chords

$$\sqrt{a}x\cos\frac{\alpha+\gamma}{2} + \sqrt{b}y\sin\frac{\alpha+\gamma}{2} - z\sqrt{-c}\cos\frac{\alpha-\gamma}{2} = 0$$

and $$\sqrt{a}x\cos\frac{\beta+\gamma}{2} + \sqrt{b}y\sin\frac{\beta+\gamma}{2} - z\sqrt{-c}\cos\frac{\beta-\gamma}{2} = 0$$

are both tangents to (1).

$$\therefore \quad a\cos^2\frac{\alpha+\gamma}{2}+b\sin^2\frac{\alpha+\gamma}{2}-c\cos^2\frac{\alpha-\gamma}{2}=0 \;\ldots\ldots\ldots\ldots\; (4),$$

$$a\cos^2\frac{\beta+\gamma}{2}+b\sin^2\frac{\beta+\gamma}{2}-c\cos^2\frac{\beta-\gamma}{2}=0 \;\ldots\ldots\ldots\ldots\; (5).$$

We want to find the envelope of the line (3). Let us then write this line $lx+my+nz=0$, where

$$l=\sqrt{a}\cos\frac{\alpha+\beta}{2}, \quad m=\sqrt{b}\sin\frac{\alpha+\beta}{2}, \quad n=-\sqrt{-c}\cos\frac{\alpha-\beta}{2}.$$

Now from (4) and (5) we see that α and β are roots of the equation in θ, viz.

$$a\cos^2\frac{\theta+\gamma}{2}+b\sin^2\frac{\theta+\gamma}{2}-c\cos^2\frac{\theta-\gamma}{2}=0,$$

that is $\quad a(1+\cos\overline{\theta+\gamma})+b(1-\cos\overline{\theta+\gamma})-c(1+\cos\overline{\theta-\gamma})=0,$

that is $\quad a+b-c+(a-b-c)\cos\theta\cos\gamma-(a-b+c)\sin\theta\sin\gamma=0,$

which, if we write

$$b+c-a=A, \quad c+a-b=B, \quad a+b-c=C,$$

becomes $\quad A\cos\theta\cos\gamma+B\sin\theta\sin\gamma=C.$

Thus we have

$$A\cos\alpha\cos\gamma+B\sin\alpha\sin\gamma=C,$$

$$A\cos\beta\cos\gamma+B\sin\beta\sin\gamma=C.$$

And from these we have at once

$$\frac{A\cos\gamma}{\cos\frac{\alpha+\beta}{2}}=\frac{B\sin\gamma}{\sin\frac{\alpha+\beta}{2}}=\frac{C}{\cos\frac{\alpha-\beta}{2}},$$

that is

$$\frac{\frac{\cos\gamma}{l}}{\frac{}{A\sqrt{a}}}=\frac{\frac{\sin\gamma}{m}}{\frac{}{B\sqrt{b}}}=-\frac{\frac{1}{n}}{\frac{}{C\sqrt{-c}}},$$

$$\therefore \quad \frac{l^2}{aA^2}+\frac{m^2}{bB^2}+\frac{n^2}{cC^2}=0.$$

Thus the envelope of the line is

$$aA^2x^2+bB^2y^2+cC^2z^2=0,$$

that is $\quad a(b+c-a)^2x^2+b(c+a-b)^2y^2+c(a+b-c)z^2=0.$

Now we have

$$\Delta=1, \quad \Theta=a+b+c, \quad \Theta'=bc+ca+ab, \quad \Delta'=abc.$$

Thus the envelope is

$$a(\Theta-2a)^2x^2+b(\Theta-2b)^2y^2+c(\Theta-2c)^2z^2=0,$$

that is

$$\Theta^2(ax^2+by^2+cz^2)-4\Theta(a^2x^2+b^2y^2+c^2z^2)+4(a^3x^2+b^3y^2+c^3z^2)=0,$$

that is $\quad \Theta^2S'-4\{a(b^2y^2+c^2z^2)+b(c^2z^2+a^2x^2)+c(a^2x^2+b^2y^2)\}=0,$

or $\quad \Theta^2S'-4\{bc(by^2+cz^2)+ca(cz^2+ax^2)+ab(ax^2+by^2)\}=0,$

that is $\Theta^2S' - 4\{bc(S'-ax^2)+ca(S'-by^2)+ab(S'-cz^2)\}=0,$

that is $\Theta^2S' - 4\Theta'S' + 4abcS' = 0,$

which we write $(\Theta^2 - 4\Theta'\Delta)S' + 4\Delta\Delta'S = 0,$

so that it is homogeneous in Δ, Δ', Θ and Θ'.

This then is the equation of the envelope for all homogeneous transformations of S and S'.

We observe that if $\Theta^2 - 4\Theta'\Delta = 0$, this reduces to the conic $S=0$. (Compare § 360.)

The following short and elegant solution of the problem is given by Salmon.

If we take PQR for triangle of reference we can write

$$S = x^2 + y^2 + z^2 - 2yz - 2zx - 2\lambda xy,$$

and $$S' = 2fyz + 2gzx + 2hxy.$$

Let $\lambda = 1 + hk$, then

$$S = x^2 + y^2 + z^2 + 2yz - 2zx - 2xy - 2hkxy,$$

$$\therefore \quad S + kS' = x^2 + y^2 + z^2 + 2(fk-1)yz + 2(gk-1)zx - 2xy.$$

Hence $S+kS'=0$ is a conic touching $z=0$, that is QR.

It can now be shewn that this conic is a *fixed* conic.

For we have

$$\Delta = 1 - 2(1+hk) - 1 - 1 - (1+hk)^2$$
$$= -4 - 4hk - h^2k^2 = -(2+hk)^2,$$
$$\Theta = 2f(1+hk+1) + 2g(1+hk+1) + 2h(1+1+hk)$$
$$= 2(f+g+h)(2+hk),$$
$$\Theta' = -(f+g+h)^2 - 2fghk,$$
$$\Delta' = 2fgh.$$

Thus $$\Theta^2 - 4\Delta\Theta' = 4\Delta\Delta'k.$$

Hence the conic $S+kS'=0$ is

$$(\Theta^2 - 4\Delta\Theta')S + 4\Delta\Delta'S = 0.$$

This being homogeneous in Δ, Δ', Θ, Θ' must be a fixed conic.

2. *The sides of a triangle touch a conic S and two of its vertices lie on another conic S', to find the locus of the third vertex.*

We can so transform our equations as that

$$S \equiv ax^2 + by^2 + cz^2,$$
$$S' \equiv x^2 + y^2 + z^2.$$

So that $\Delta = abc, \quad \Theta = \Sigma bc, \quad \Theta' = \Sigma a, \quad \Delta' = 1.$

Now reciprocate with respect to S'; we have two conics

$S_1 \equiv bcx^2 + cay^2 + abz^2$, the reciprocal of S,

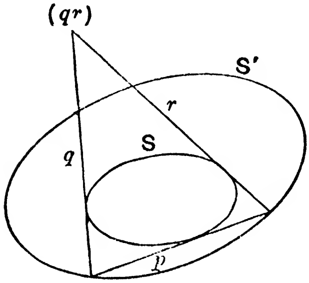
and $S' \equiv x^2 + y^2 + z^2,$

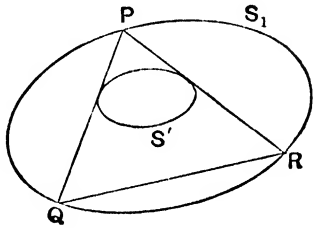
such that tangents from P (the reciprocal of p) to S' meet S_1 in Q and R, the reciprocals of q and r. And by the previous example the envelope of QR is

$$(\Theta_1'^2-4\Theta_1\Delta_1')\,S_1+4\Delta_1\Delta_1'S'=0,$$

where $$\Delta_1=a^2b^2c^2=\Delta^2,\quad \Delta_1'=1=\Delta'^2,$$

$$\Theta_1=abc\,(a+b+c)=\Delta\Theta',\quad \Theta_1'=bc+ca+ab=\Theta=\Delta'\Theta.$$





Thus the envelope of QR is

$$(\Delta'^2\Theta^2-4\Delta\Theta'\Delta'^2)\,(bcx^2+cay^2+abz^2)+4\Delta^2\Delta'^2\,(x^2+y^2+z^2)=0,$$

that is $bcx^2+cay^2+abz^2+\lambda\,(x^2+y^2+z^2)=0$, where $\lambda=\dfrac{4\Delta^2}{\Theta^2-4\Delta\Theta'}$.

The locus of (qr) is the polar reciprocal of this with respect to S', which is

$$(bc+\lambda)\,(ca+\lambda)\,x^2+(ca+\lambda)\,(ab+\lambda)\,y^2+(ab+\lambda)\,(bc+\lambda)\,z^2=0,$$

that is $$abcS+\lambda F+\lambda^2S'=0,$$

which is $$\Delta S+\lambda F+\lambda^2\Delta'S'=0.$$

We now substitute for λ, and get

$$\Delta S\,(\Theta^2-4\Delta\Theta')^2+4\Delta^2F(\Theta^2-4\Delta\Theta')+16\Delta^4\Delta'S'=0.$$

On dividing by Δ we have

$$(\Theta^2-4\Delta\Theta')^2\,S+4\Delta\,(\Theta^2-4\Delta\Theta')\,F+16\Delta^3\Delta'S'=0$$

398. The Φ conic.

As the locus of points the pairs of tangents from which to two conics form a harmonic pencil is a conic, it follows by reciprocation that the envelope of lines cutting two conics in two pairs of points forming a harmonic range is a conic also.

Let us now find the equation of this conic for the two conics

$$S\equiv(a,\,b,\,c,\,f,\,g,\,h)\,(x,\,y,\,z)^2=0\,\ldots\ldots\ldots\ldots\ldots(1),$$

$$S'\equiv(a',\,b',\,c',\,f',\,g',\,h')\,(x,\,y,\,z)^2=0\ \ldots\ldots\ldots(2).$$

If we reciprocate these two conics with respect to

$$x^2 + y^2 + z^2 = 0,$$

we get $$S_1 \equiv (A, B, C, F, G, H)(x, y, z)^2 = 0 \quad \text{.........(3)},$$

$$S_2 \equiv (A', B', C', F', G', H')(x, y, z)^2 = 0 \quad \text{......(4)}.$$

Now clearly if $lx + my + nz = 0$ be a line which cuts (1) in two points harmonically conjugate with the two points in which it cuts (2), then (l, m, n) will be a point on the **F** conic of (3) and (4).

But as the minors of A, B, C etc. in the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}$$

are proportional to a, b, c etc., being in fact

$$a\Delta, b\Delta, c\Delta \text{ etc....(§ 372)},$$

and similarly the minors of A', B', C' etc. in their determinant are $a'\Delta', b'\Delta'$ etc., therefore the **F** conic of (3) and (4) is

$$(bc' + b'c - 2ff')x^2 + \text{two similar terms}$$

$$+ 2(gh' + g'h - af' - a'f)yz + \text{two similar terms} = 0.$$

As then (l, m, n) lies on this we have

$$(bc' + b'c - 2ff')l^2 + \text{etc.} + 2(gh' + g'h - af' - a'f)mn + \text{etc.} = 0.$$

That is $\Phi = 0$, where Φ is as defined in § 379.

This then is the tangential equation of the conic which is the envelope of lines cutting (1) and (2) in pairs of points harmonically conjugate.

It is convenient to speak of this conic as the Φ conic of (1) and (2).

399. Point equation of Φ conic.

To find the point equation for the Φ conic of

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0 \quad \text{...............(1)},$$

$$S' \equiv (a', b', c', f', g', h')(x, y, z)^2 = 0 \quad \text{...........(2)}.$$

We transform these conics into

$$S \equiv \alpha X^2 + \beta Y^2 + \gamma Z^2 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$S' \equiv X^2 + Y^2 + Z^2 \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

For (3) and (4)

$$\Delta = \alpha\beta\gamma, \qquad \Theta = \beta\gamma + \gamma\alpha + \alpha\beta,$$

$$\Theta' = \alpha + \beta + \gamma, \qquad \Delta' = 1,$$

and $$\mathbf{F} = \alpha(\beta+\gamma)X^2 + \beta(\gamma+\alpha)Y^2 + \gamma(\alpha+\beta)Z^2.$$

Now the Φ conic of (3) and (4) is

$$(\beta+\gamma)l^2 + (\gamma+\alpha)m^2 + (\alpha+\beta)n^2 = 0,$$

and the corresponding point equation is

$$(\gamma+\alpha)(\alpha+\beta)X^2 + (\alpha+\beta)(\beta+\gamma)Y^2 + (\beta+\gamma)(\gamma+\alpha)Z^2 = 0.$$

The coefficient of X^2 is

$$\begin{aligned}\alpha(\alpha+\beta+\gamma)+\beta\gamma &\\ &= \alpha(\alpha+\beta+\gamma) + (\beta\gamma+\gamma\alpha+\alpha\beta) - \alpha(\beta+\gamma)\\ &= \alpha\Theta' + \Theta - \alpha(\beta+\gamma).\end{aligned}$$

Thus the point Φ equation of (3) and (4) is

$$\begin{aligned}\Theta'(\alpha X^2+\beta Y^2+\gamma Z^2) + \Theta(X^2+Y^2+Z^2)&\\ - \{\alpha(\beta+\gamma)X^2 + \beta(\gamma+\alpha)Y^2 + \gamma(\alpha+\beta)Z^2\} &= 0,\end{aligned}$$

that is $$\Theta' S + \Theta S' - \mathbf{F} = 0.$$

As this is homogeneous in Θ, Θ', $\mathbf{F}$, it follows that it is the point equation of (1) and (2) when Θ, Θ', $\mathbf{F}$ refer to those two.

400. Contravariants.

The expression Φ which stands for

$$(bc' + b'c - 2ff')l^2 + \text{etc.}, + 2(gh' + g'h - af' - a'f)mn + \text{etc.}$$

is called a 'contravariant' for the conics

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0,$$

$$S' \equiv (a', b', c', f', g', h')(x, y, z)^2 = 0.$$

A contravariant differs from a covariant in that it is a function of l, m, n and the coefficients, not of x, y, z and the coefficients. A contravariant equated to zero will give the *tangential equation* of some locus geometrically related to the

two conics; while a covariant equated to zero gives the *point equation* of some locus geometrically related to them.

The analytical distinction between a covariant and a contravariant can be seen in the following way.

If the equations of the conics be transformed by substitutions of the form

$$\left.\begin{aligned} x &= \lambda_1 X + \mu_1 Y + \nu_1 Z \\ y &= \lambda_2 X + \mu_2 Y + \nu_2 Z \\ z &= \lambda_3 X + \mu_3 Y + \nu_3 Z \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

the line $lx + my + nz = 0$ will be transformed into

$$l'X + m'Y + n'Z = 0,$$

where

$$\left.\begin{aligned} l' &= \lambda_1 l + \lambda_2 m + \lambda_3 n \\ m' &= \mu_1 l + \mu_2 m + \mu_3 n \\ n' &= \nu_1 l + \nu_2 m + \nu_3 n \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

Now if $\qquad f(a, b, c \ldots a', b', c' \ldots l, m, n)$

be a contravariant,

$$f(a_1, b_1, c_1 \ldots a_1', b_1', c_1' \ldots l', m', n')$$

will be

$$\epsilon^k f(a, b, c \ldots a_1, b_1, c_1 \ldots l, m, n),$$

where a_1, b_1, c_1 etc. are the new coefficients of X^2, Y^2 etc. in the transformed equation and ϵ is the determinant

$$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \lambda_3, & \mu_3, & \nu_3 \end{vmatrix}.$$

We observe then that the l, m, n are transformed into l', m', n' by the substitutions (B), whereas x, y, z are transformed into X, Y, Z by the substitutions (A).

We can then define a contravariant in this way:

A homogeneous function of l, m, n and of the coefficients of two conics S and S' is called contravariant when, on the equation of the conic being transformed by the linear substitutions

$$\left.\begin{aligned} x &= \lambda_1 X + \mu_1 Y + \nu_1 Z \\ y &= \lambda_2 X + \mu_2 Y + \nu_2 Z \\ z &= \lambda_3 X + \mu_3 Y + \nu_3 Z \end{aligned}\right\},$$

and the l, m, n being transformed to l', m', n' by the substitutions

$$\left.\begin{aligned} l' &= \lambda_1 l + \lambda_2 m + \lambda_3 n \\ m' &= \mu_1 l + \mu_2 m + \mu_3 n \\ n' &= \nu_1 l + \nu_2 m + \nu_3 n \end{aligned}\right\},$$

the function which is related to the new coefficients of S and S' and to l', m', n' exactly as was the original function to the old coefficients of S and S' and to l, m, n bears a constant ratio to it, viz. ϵ^k where ϵ^k denotes some power of the determinant of transformation.

401. The Jacobian of three conics.

If we have three conics

$$\begin{aligned} S &\equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \\ S' &\equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0, \\ S'' &\equiv a''x^2 + b''y^2 + \ldots = 0, \end{aligned}$$

and we write down the equations of the polars of these with respect to the point (x_1, y_1, z_1), we obtain

$$\begin{aligned} (ax_1 + hy_1 + gz_1)\,x + (hx_1 + by_1 + fz_1)\,y + (gx_1 + fy_1 + cz_1)\,z &= 0, \\ (a'x_1 + h'y_1 + g'z_1)\,x + \ldots &= 0, \\ (a''x_1 + h''y_1 + g''z_1)\,x + \ldots &= 0. \end{aligned}$$

The condition that these three lines should be concurrent is

$$\begin{vmatrix} ax_1 + hy_1 + gz_1, & hx_1 + by_1 + fz_1, & gx_1 + fz_1 + cz_1 \\ a'x_1 + h'y_1 + g'z_1, & h'x_1 + b'y_1 + f'z_1, & g'x_1 + f'y_1 + c'z_1 \\ a''x_1 + h''y_1 + g''z_1, & h''x_1 + b''y_1 + f''z_1, & g''x_1 + f''y_1 + c''z_1 \end{vmatrix} = 0.$$

Hence the locus of points whose polars with respect to the three conics are concurrent is the cubic curve

$$\begin{vmatrix} ax + hy + gz, & hy + by + fz, & gx + fy + cz \\ a'x + h'y + g'z, & h'x + b'y + f'z, & g'x + f'y + c'z \\ a''x + h''y + g''z, & h''x + b''y + f''z, & g''x + f''y + c''z \end{vmatrix} = 0.$$

The expression on the left-hand side is known as the *Jacobian* of the three conics. In the notation of the differential calculus the equation can be written

$$\begin{vmatrix} S_x, & S_y, & S_z \\ S'_x, & S'_y, & S'_z \\ S''_x, & S''_y, & S''_z \end{vmatrix} = 0,$$

where
$$S_x \equiv \frac{\partial S}{\partial x}.$$

402. Special cases of the Jacobian.

In the special case where the three conics have a common self-conjugate triangle we can refer them to this triangle and write

$$S \equiv ax^2 + by^2 + cz^2,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2,$$
$$S'' \equiv a''y^2 + b''y^2 + c''z^2.$$

The Jacobian curve, or locus of points whose polars with respect to the conics are concurrent, is then

$$\begin{vmatrix} ax, & by, & cz \\ a'x, & b'y, & c'z \\ a''x, & b''y, & c''z \end{vmatrix} = 0,$$

that is
$$xyz = 0,$$

which represents the three lines on which lie the sides of the triangle of reference.

403. Should it happen that the three conics all pass through the same two points, A and B say, then it is clear that the line AB is a part of the locus. For if P be any point on AB and we take Q in the same line so that

$$(AB, PQ) = -1.$$

the polars of P with respect to all three conics will be concurrent in Q. Thus all points in the line AB belong to the locus.

In this case then the Jacobian curve will be a line and a conic.

This will always happen when the three conics are all circles; for all circles have the two circular points at infinity

common. Thus the Jacobian for three circles will be the line at infinity and a conic.

We can now shew that the conic in this case is the circle which cuts the three circles orthogonally.

For the circle which cuts the three circles orthogonally is the conic through the six limiting points of the circles taken in pairs, and these limiting points belong to the locus of points the polars of which for the three circles are concurrent. This indeed follows from the fact that the polars of a limiting point with respect to two circles are the same line.

404. The Jacobian will still reduce to a line and a conic if the three conics all touch at a common point, for the common tangent there is obviously a part of the locus.

If the three conics have double contact at the same two points, then their Jacobian must vanish identically, for they have an infinite number of common self-conjugate triangles and all points on the sides of these satisfy the geometrical property that determines the Jacobian (§ 401).

Examples. 1. If three conics have four-point contact at the same point, shew that their Jacobian is identically zero.

2. If three conics have three-point contact at the same point, shew that their Jacobian reduces to the cube of their common tangent at that point.

3. If three conics have three-point contact at the same point, and a second point common to all three, their Jacobian is identically zero.

405. The cubic covariant of two conics.

If we form the Jacobian of two conics S and S' and their **F** conic, it is clear that this will give a covariant for the two conics, which is in the third degree in the variables.

That the Jacobian is a covariant is clear from the fact that when equated to zero it gives a geometrical locus connected with the two conics, viz. the locus of points the polar of which with respect to S and S' and **F**, should be concurrent.

From what has been already said about the Jacobian of three conics we have the following properties of the cubic covariant in special cases:

(1) If S and S' do not touch, in which case S, S' and $\mathbf{F}$ have a common self-conjugate triangle, the Jacobian (denoted by the letter J) is the three sides of this common self-polar triangle.

(2) If S and S' have simple contact, J will be the product of their common tangent and a quadratic function of x, y, z.

(3) If S and S' have double contact, in which case $\mathbf{F}$ has double contact with them, J will vanish identically.

(4) If S and S' have three-point contact, in which case $\mathbf{F}$ has also three-point contact with them, J will reduce to the cube of their common tangent.

(5) If S and S' have four-point contact, J will again vanish identically.

406. Jacobian of S, S′ and their Φ conic.

The Jacobian of two conics S and S' and their Φ conic will be the same locus as the Jacobian of S, S' and $\mathbf{F}$, for as we have seen (§ 399) the Φ conic is

$$\Theta' S + \Theta S' - \mathbf{F} = 0,$$

and the left side is a linear function of S, S' and $\mathbf{F}$.

407. The cubic contravariant.

It is clear from the principles of reciprocation that there must be a cubic contravariant of two conics S and S' to correspond with the cubic covariant J. This (which is denoted by Γ) equated to zero will give the tangential equation of the envelope of lines whose poles with regard to S, S' and their Φ conic are collinear.

Γ will be the Jacobian with respect to l, m, n of Σ, Σ' and Φ, $\Sigma = 0$, $\Sigma' = 0$ being the tangential equations of the two conics.

That this is so is seen from the fact that $\Sigma = 0$, $\Sigma' = 0$ when x, y, z are written, for l, m, n are the equations of the polar reciprocals of $S = 0$, $S' = 0$ with respect to the conic

$$x^2 + y^2 + z^2 = 0,$$

and $\Phi = 0$ when x, y, z are written, for l, m, n is the locus of points the tangents from which to these two conics, which are the reciprocals of S and S', are harmonically conjugate.

Thus $\Sigma = 0$, $\Sigma' = 0$, $\Phi = 0$ when in them x, y, z are written, for l, m, n correspond to the conics $S = 0$, $S' = 0$, $\mathbf{F} = 0$.

Thus the Jacobian of Σ, Σ', Φ regarded as functions of x, y, z will be the locus of points the polars of which with regard to these conics are concurrent.

Now when we replace x, y, z by l, m, n again, we have a tangential equation which represents a curve possessing the property with regard to S, S' and $\mathbf{F}$ which is the reciprocal of that possessed by Σ_{xyz}, Σ'_{xyz}, Φ_{xyz}.

Hence the Jacobian of Σ, Σ' and Φ with respect to l, m, n equated to zero will be the tangential equation of the envelope of lines whose poles with respect to the three conics whose tangential equations are $\Sigma = 0$, $\Sigma' = 0$, $\Phi = 0$ are collinear.

Note. In the examples which follow F is written for $\mathbf{F}$, there being no fear that this F will be confused with the minor of f in the determinant Δ.

EXAMPLES.

1. If for two conics $S=0$, $S'=0$, the invariant $\Theta=0$, their Φ conic is the polar reciprocal of S with respect to S'.

2. The equation of the four tangents to any conic $S=0$ at the points where it is cut by $S'=0$ is

$$(\Theta S-\Delta S')^2=4\Delta S(\Theta' S-F).$$

3. Prove that the Φ conic for two circles which cut orthogonally degenerates into their two centres.

4. The envelope of the lines cutting two conics S and S' in pairs of points harmonically conjugate degenerates into two points if

$$\Theta\Theta'=\Delta\Delta'.$$

Shew too that this is the condition that the F conic should degenerate into two lines.

5. Two equal rectangular hyperbolas are so placed that the transverse axis of the one is in the same straight line with the conjugate axis of the other. Prove that the straight line which is cut by them harmonically envelopes a hyperbola of eccentricity $\sqrt{\frac{3}{2}}$.

6. If $F=0$ be the locus of points from which tangents to the conics S and S' are harmonically conjugate, and if $F'=0$ be the envelope of lines divided harmonically by the two conics, then if $F=0$, $F'=0$ be such that triangles can be inscribed in F self-conjugate with respect to F',

$$\Theta\Theta'+3\Delta\Delta'=0.$$

7. The locus of points from which pairs of tangents to the two conics $S=0$, $S'=0$ are harmonically conjugate is denoted by $F=0$, and the envelope of lines divided harmonically by the conics is $F'=0$. If $F=0$, $F'=0$ have double contact then S and S' have double contact or $\Delta\Theta'^2=\Delta'\Theta^2$.

8. The Φ conic of $S=0$, $S''=0$ is $S'=0$, and the polar reciprocal of $S=0$ with respect to $S''=0$ is $kS+S'=0$, shew that

$$k=\tfrac{1}{4}(\Theta'^2-4\Delta'\Theta)/\Delta\Delta',$$

where Δ, Θ etc. refer to S and S'.

9. Shew that the locus of points, the tangents from which to two orthogonal circles form a harmonic pencil, is composed of the chords of contact of their common tangents.

10. Shew that the point equation of $k\Sigma + \Phi = 0$, where Σ and Φ have their usual meaning in relation to two conics S and S', is

$$k^2\Delta S + k(\Theta S + \Delta S') + (S\Theta' + S'\Theta - F) = 0,$$

and interpret the equation

$$(\Theta S + \Delta S')^2 - 4\Delta S(S\Theta' + S'\Theta - F) = 0.$$

11. Shew that if the director circle of the conic which passes through the points of contact of the common tangents of two circles is coaxial with these circles, then one of the limiting points is midway between the centres of the two circles.

12. Two conics $S=0$, $S'=0$ are such that triangles ABC can be inscribed in S' so that their sides touch S at A', B', C'. Prove that, as the triangle changes its position, the point of concurrence of the lines AA', BB', CC' traces out the locus

$$2\Theta S = 3\Delta S'.$$

13. If TP and TQ be tangents to a conic S, and TP', TQ' tangents from the same point T to another conic S' and $T(PP'QQ') = k$ a constant, prove that the locus of T is

$$\frac{F^2}{4\Delta\Delta' SS'} = \left(\frac{k+1}{k-1}\right)^2$$

and interpret the cases where $k=0$, $k=1$, $k=-1$.

14. If a conic S and a rectangular hyperbola S' are so related that the centre of S' lies on the director circle of S, then the conic F is a rectangular hyperbola.

15. An ellipse of eccentricity e and a coaxial hyperbola of eccentricity e' are such that the eight points of contact of common tangents lie on two straight lines. Prove that either a pair of collinear axes are equal or

$$e'^2 + e^2 = 2 \text{ or } (1-e^2)(e'^2-1) = 1.$$

16. Prove that the F conic of two parabolas is in general a hyperbola but that in special cases it may be a parabola.

The F conic of the two parabolas

$$(a, b, c, f, g, h)(x, y, z)^2 = 0,$$

$$(a', b', c', f', g', h')(x, y, z)^2 = 0,$$

will be a parabola if

$$(gh - af)(h'f' - b'g') = (g'h' - a'f')(hf - bg).$$

17. Prove that

$$J^2 = F^3 - (\Theta' S + \Theta S') F^2 + (\Delta'\Theta S^2 + \Delta\Theta' S'^2) F + (\Theta\Theta' - 3\Delta\Delta') FSS'$$
$$- (\Theta^2 - 2\Delta\Theta')\Delta' S^2 S' - (\Theta'^2 - 2\Delta'\Theta)\Delta SS'^2 - \Delta\Delta'^2 S^3 - \Delta^2\Delta' S'^3.$$

[Take $\qquad S = ax^2 + by^2 + cz^2,\ S' = x^2 + y^2 + z^2,$

so that $\qquad F = a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2,$

and $\qquad J = (b-c)(c-a)(a-b)xyz.$]